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On the Cauchy- Schwarz Inequality for Vectors

in the inner Product spaces with Applications

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Abstract: In this paper, eleven different proofs for Cauchy Schwarz inequality are considered on the inner product spaces, and specific teqnique of the inequality was used to establish various proofs of some new related inequalities are obtained, Which contributed to the establishment of different mathematical inequalities, notably the application, which shows multiple relationships on the sides of the triangle.

Keyword: Cauchy-Schwarz inequality, inner product space, norm of vector, scalar multiple, unit vector, orthogonal decomposition, mathematical induction.

1- Introduction:

In mathematics, the Cauchy-Schwarz inequality, is the one of the most used inequality in mathematics [1] ,the applications of this famous inequality include linear algebra, probability theory ,as well as important topics in physics and engineering[2] .Many proofs of the Cauchy-Schwarz inequality in complex spaces are presented by Volker, see[3],also Hui - hua and Shanhe introduced some proofs of in the same theorem, in the sense of series , see [4].In this paper we show some different proofs of this theorem. S.S. DRAGOMIR obtain some new Schwarz related inequalities in inner product spaces over the real or complex number field. Applications for the generalized triangle inequality[5] , KOSTADIN and RISTO obtain some new proofs of The Cauchy-Schwarz inequality for the general type of n-inner product and some applications are given[6].

Problem statement and objectives:

As we know, those who concerned in mathematics faced a lot of problems and how could solve it, many students have shown that the issue of comparisons in mathematics are the most complex, that left negative results, which has prompted us to find an establish a work that would frame the solution of these sort of problems by using Cauchy- Schwarz Inequality for Vectors. Therefore, this paper targets to improve the understanding of the Cauchy- Schwarz Inequality for Vectors, so as to explore the important application of this theory.

Methods:

our paper classified as literature review which is acritical analysis of published knowledge through summary and classification, so we use various resources such as journal articles, books, and websites. We use the descriptive analytical method in order to answer the following questions:

1- What is the Cauchy- Schwarz Inequality for Vectors in the inner Product spaces?
2- How we can improve our understanding to this inequality?
3- What is the important application of Cauchy- Schwarz Inequality for Vectors in the inner Product spaces?

Definitions serve the search:

Definition1.1: An inner product space V is a vector space over the field F of real numbers R, equipped with a mapping $\langle\cdot.\cdot\rangle: \mathrm{V} \times \mathrm{V} \to \mathrm{F}$ which satisfies the following properties: [2] .

1) Linearity: $\langle\vec{u}\ + \vec{v}.\vec{w}\rangle = \langle\vec{u}.\vec{w}\rangle + \langle\vec{v}.\vec{w}\rangle$,
2) $\langle\alpha\vec{u}\,.\,\vec{v}\ \rangle = \alpha\langle\vec{u}\,.\,\vec{v}\rangle$,
3) Symmetry: $\langle\vec{u}\,.\,\vec{v}\rangle = \langle\vec{v}.\vec{u}\ \rangle$,
4) Positivity: $\langle\vec{v}.\vec{v}\rangle \geq 0$, and $\langle\vec{v}.\vec{v}\rangle = 0$, iff $\vec{v} = 0$, for all $\vec{u}.\vec{v}.\vec{w} \in \mathrm{V}$, and $\alpha \in \mathrm{F}$.

Definition 1.2: The norm of vector spaces is the analogue of the length .it is formally defined as follows. Let V be a vector space over F, equipped with a mapping
$\|\cdot\|: \mathrm{V} \to \mathrm{F}$, which satisfies the following properties. [2].

(1) $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$, if $\vec{v} = 0$,
(2) $\|\alpha\vec{v}\| = |\alpha|\|\vec{v}\|$,
(3) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (Triangle inequality), for all $\vec{u}.\vec{v}\ \in \mathrm{V}$ and $\alpha \in \mathrm{F}$.

Vector Form of the Cauchy-Schwarz inequality 1.3:

the Cauchy–Schwarz inequality states that for all vectors $\vec{u}$ and $\vec{v}$ of an inner product space it is true that:

$$|\langle\vec{v}.\vec{v}\rangle| = \|\vec{v}\| \cdot \|\vec{v}\|$$

Furthermore, the inequality holds if and only if u and v are linearly dependent, i.e. are scalar multiple of each other. this inequality may be stated as follows:

$|\langle\vec{u}.\vec{v}\rangle|^2 = \langle\vec{u}.\vec{u}\rangle \cdot \langle\vec{v}.\vec{v}\rangle$, or

$|\mathrm{u}_1\mathrm{v}_1.\mathrm{u}_2\mathrm{v}_2.\mathrm{u}_3\mathrm{v}_3.\dots.\mathrm{u}_n\mathrm{v}_n|^2 \leq (\mathrm{u}_1.\mathrm{u}_2.\mathrm{u}_3.\dots.\mathrm{u}_n)^2(\mathrm{v}_1.\mathrm{v}_2.\mathrm{v}_3.\dots.\mathrm{v}_n)^2$, (see [4])

Also, it can be stated as the following determinant:

$$\begin{vmatrix} \langle u_1.v_1\rangle & \langle u_1.v_2\rangle & \langle u_1.v_3\rangle & \cdots & \langle u_1.v_n\rangle \\ \langle u_2.v_1\rangle & \langle u_2.v_2\rangle & \langle u_2.v_3\rangle & \cdots & \langle u_2.v_n\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle u_n.v_1\rangle & \langle u_n.v_2\rangle & \langle u_n.v_3\rangle & \ldots & \langle u_n.v_n\rangle \end{vmatrix} \geq 0.$$

For all sequences of real numbers a_i and b_i, we have:

$$\left(\sum_{i=1}^{n} a_i \cdot b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

2- proofs of the inequality:

In this paper, we show different proof of the inequality as follows:

Proof 2.1: using Pythagorean theorem: (see [7])

Part I: if $\vec{v} = \vec{0}$, $\langle \vec{u}.\vec{v}\rangle = \langle \vec{u}.\vec{0}\rangle = 0$ and $\|\vec{u}\| \cdot \|\vec{v}\| = \|\vec{u}\| \cdot \|\vec{0}\| = 0$, then both sides of the desired inequality equal zero, so the inequality hold. Thus, we can assume that $\vec{v} \neq 0 \Rightarrow \langle \vec{u}.\vec{v}\rangle \neq 0$,then by using the diagram, $\vec{u}$ can be written as: $\vec{u} = \overrightarrow{u_1} + \vec{p}$

Then, $\vec{p} = \vec{u} - \overrightarrow{u_1} = \vec{u} - \alpha\vec{v}$, α be any scalar , we have $\vec{p} \perp \overrightarrow{u_1}$, then $\vec{u} - \alpha\vec{v} \perp \vec{v}$, so we get:

$$\begin{aligned} 0 = \langle \vec{u} - \alpha\vec{v}.\vec{v}\rangle &= \langle \vec{u}.\vec{v}\rangle - \langle \alpha\vec{v}.\vec{v}\rangle \\ &= \langle \vec{u}.\vec{v}\rangle - \alpha\langle \vec{v}.\vec{v}\rangle \\ &= \langle \vec{u}.\vec{v}\rangle - \alpha\|\vec{v}\|^2 \end{aligned}$$

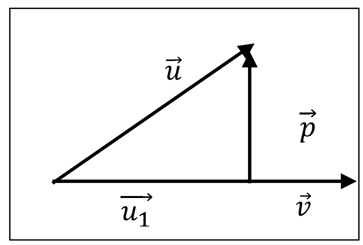

Figure 1pythagrean theorem for inner product

By solving for α , then we have:

So that $\overrightarrow{u_1} = \alpha\vec{v} = \frac{\langle \vec{u}.\vec{v}\rangle}{\|\vec{v}\|^2} \cdot \vec{v}$, now, we can apply the Pythagorean theorem for inner product spaces, since $\vec{u}$ is the sum of $\overrightarrow{u_1}$ and $\vec{p}$, where $\vec{p} \perp \overrightarrow{u_1}$, then

$$\begin{aligned} \|\vec{u}\|^2 &= \|\overrightarrow{u_1}\|^2 + \|\vec{p}\|^2 \\ &= \left\|\frac{\langle \vec{u}.\vec{v}\rangle}{\|\vec{v}\|^2} \cdot \vec{v}\right\|^2 + \|\vec{p}\|^2 \end{aligned}$$

$\|\vec{u}\|^2 = \frac{\|\langle \vec{u}.\vec{v}\rangle\|^2}{\|\vec{v}\|^2} + \|\vec{p}\|^2$," by delete $\|\vec{p}\|^2$ from the left side" Then $\|\vec{u}\|^2 \geq \frac{\|\langle \vec{u}.\vec{v}\rangle\|^2}{\|\vec{v}\|^2}$

We now multiply both sides by $\|\vec{v}\|^2$, and then take the square root, we get:

$$|\langle \vec{u}.\vec{v}\rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$$

Part II: (see [7]).

Suppose that $\vec{u}$ is a scalar multiple of $\vec{v}$,then $\vec{u} = \alpha\vec{v}$, α is any scalar, so

$$\begin{aligned} |\langle \vec{u}.\vec{v}\rangle| = |\langle \alpha\vec{v}.\vec{v}\rangle| &= \alpha|\langle \vec{v}.\vec{v}\rangle| \\ &= \alpha\|\vec{v}\|^2 \\ &= \alpha\|\vec{v}\| \cdot \|\vec{v}\| \end{aligned}$$

$= \|\alpha\vec{v}\| \cdot \|\vec{v}\|$

$= \|\vec{u}\| \cdot \|\vec{v}\|$, hence the equality hold.

Conversely, suppose that: $|\langle\vec{u}.\vec{v}\rangle| = \|\vec{u}\| \cdot \|\vec{v}\|$, and we have from the upper proof:

$\|\vec{u}\|^2 = \left\|\frac{\langle\vec{u}.\vec{v}\rangle}{\|\vec{v}\|^2} \cdot \vec{v}\right\|^2 + \|\mathrm{p}\|^2$, by substituting we get:

$\|\vec{u}\|^2 = \frac{\|\vec{u}\|^2 \cdot \|\vec{v}\|^2}{\cdot\|\vec{v}\|^2} + \|\vec{p}\|^2$

$\|\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{p}\|^2$

Which implies that $\|\vec{p}\|^2 = 0$,and so $\|\vec{p}\| = 0$, But $\vec{u} = \frac{\langle\vec{u}.\vec{v}\rangle}{\|\vec{v}\|^2} \cdot \vec{v} + \vec{p} \Rightarrow \vec{u} = \frac{\langle\vec{u}.\vec{v}\rangle}{\|\vec{v}\|^2} \cdot \vec{v}$, then $\vec{u}$ is multiple scalar of $\vec{v}$ .Therefore $\vec{u}$ is linearly independent of $\vec{v}$, hence the inequality hold.

Proof 2.2: using the inner product: (see [1]).

Part I: if $\vec{v} = 0$,then the equality holds trivially, so we assume that $\vec{v} \neq 0$,then for any scalar α,we have:

$$0 \le \|\vec{u} - \alpha\vec{v}\|^2 = \langle\vec{u} - \alpha\vec{v}.\vec{u} - \alpha\vec{v}\rangle$$
$$= \|\vec{u}\|^2 - \overline{\alpha}\langle\vec{u}.\vec{v}\rangle - \alpha\langle\vec{v}.\vec{u}\rangle + \alpha^2\|\vec{v}\|^2$$

In particularly, if we choose $\alpha = \frac{\langle\vec{u}.\vec{v}\rangle}{\langle\vec{v}.\vec{v}\rangle}$, from orthogonality, this implies that:

$\overline{\alpha} = \frac{\overline{\langle\vec{u}.\vec{v}\rangle}}{\langle\vec{v}.\vec{v}\rangle} = \frac{\langle\vec{u}.\vec{v}\rangle}{\langle\vec{v}.\vec{v}\rangle}$, then we have:

$$0 \le \|\vec{v}\|^2 - \frac{\langle\vec{u}.\vec{v}\rangle}{\langle\vec{v}.\vec{v}\rangle}\langle\vec{u}.\vec{v}\rangle - \frac{\langle\vec{u}.\vec{v}\rangle}{\langle\vec{v}.\vec{v}\rangle}\langle\vec{v}.\vec{u}\rangle + \frac{|\langle\vec{u}.\vec{v}\rangle|^2}{\|\vec{v}\|^4}\|\vec{v}\|^2$$
$$0 \le \|\vec{u}\|^2 - \frac{|\langle\vec{u}.\vec{v}\rangle|^2}{\|\vec{v}\|^2} - \frac{|\langle\vec{u}.\vec{v}\rangle|^2}{\|\vec{v}\|^2} + \frac{|\langle\vec{u}.\vec{v}\rangle|^2}{\|\vec{v}\|^2}$$
$$0 \le \|\vec{u}\|^2 - \frac{|\langle\vec{u}.\vec{v}\rangle|^2}{\|\vec{v}\|^2},$$

We now multiply both sides by $\|\vec{v}\|^2$, and then take the square root, we get:

$$|\langle\vec{u}.\vec{v}\rangle| \le \|\mathrm{u}\| \cdot \|\vec{v}\|$$

Part II: (see [1]). If $|\langle\vec{u}.\vec{v}\rangle| \le \|\vec{u}\| \cdot \|\vec{v}\|$ then we can choose $\alpha = 1$, such that $\langle\mathrm{u}.\vec{v}\rangle = \alpha\langle\vec{v}.\vec{v}\rangle$, then we have $\|\vec{v}\|^2 = \langle\vec{u}.\mathrm{v}\rangle$:

$$0 \le \left\|\|\vec{u}\|\vec{v} - \alpha\|\vec{v}\| \cdot \vec{u}\right\|^2 = \langle\|\vec{u}\|\vec{v} - \alpha\|\vec{v}\| \cdot \vec{u}.\|\vec{u}\|\vec{v} - \alpha\|\vec{v}\| \cdot \vec{u}\rangle$$
$$= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{v}\|^2\|\vec{u}\|^2 - \|\vec{v}\|^2\|\vec{u}\|^2$$
$$= 0.$$

Then we must have, $\|\vec{u}\|\vec{v} - \alpha\|\vec{v}\| \cdot \vec{u} = 0$

$$\|\vec{u}\|\vec{v} = \alpha\|\vec{v}\| \cdot \vec{u}$$
$$\vec{v} = \frac{\alpha\|\vec{v}\| \cdot \vec{u}}{\|\vec{u}\|},$$

So, $\vec{u}$ and $\vec{v}$ are linearly dependent.

If we take $\vec{v} = \beta u$, β be any real number.

$$0 \le |\langle \vec{u}. \beta u \rangle|^2 = \langle \vec{u}. \beta\vec{u} \rangle \cdot \langle \vec{u}. \beta\vec{u} \rangle$$
$$= \langle \vec{u}. \beta\vec{u} \rangle \cdot \langle \beta\vec{u}. \vec{u} \rangle \cdot$$
$$= \beta\langle \vec{u}. \beta\vec{u} \rangle \cdot \langle \vec{u}. \vec{u} \rangle \cdot$$
$$= \langle \beta\vec{u}. \beta\vec{u} \rangle \cdot \langle \vec{u}. \vec{u} \rangle \cdot$$
$$= \|\vec{u}\|^2 \cdot \|\beta\vec{u}\|^2 \cdot$$
$$= \|\vec{u}\|^2 \cdot \|\vec{u}\|^2.$$

By taking square root to both sides we get:

$|\langle \vec{u}. \vec{v} \rangle| = \|\vec{u}\| \cdot \|\vec{v}\|$, hence the proof follows.

Proof 2.3: by using a quadratic function: (see [8], [11])

Part I: consider the function $p(t): R \to R$, such that:

$$p(t) = \|t\vec{v} - \vec{u}\|^2$$
$$= \langle t\vec{v} - \vec{u}. tv - \vec{u} \rangle$$
$$= \langle t\vec{v}. t\vec{v} \rangle - \langle t\vec{v}. \vec{u} \rangle - \langle \vec{u}. t\vec{v} \rangle + \langle \vec{u}. \vec{u} \rangle$$
$$= t^2\|\vec{v}\|^2 - t\langle \vec{v}. \vec{u} \rangle - \bar{t}\langle \vec{u}. \vec{v} \rangle + \|\vec{u}\|^2$$
$$= t^2\|\vec{v}\|^2 - 2t\langle \vec{u}. v \rangle + \|\vec{u}\|^2.$$

Which is a quadratic function,

Let a = $\|\vec{v}\|^2$, b = $2\langle \vec{u}. \vec{v} \rangle$, c = $\|\vec{u}\|^2$, then we have $p(t) = at^2 - bt + c$

Assume that: $t = \frac{-b}{2a}$, then $p\left(\frac{-b}{2a}\right) = a\left(\frac{-b}{2a}\right)^2 - b\left(\frac{-b}{2a}\right) + c$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$
$$= \frac{b^2 - 2b^2}{4a} + c$$
$$= \frac{-b^2}{4a} + c, \text{ but } p\left(\frac{-b}{2a}\right) \ge 0$$

Forget the $p(t)$ function side (LHS), Then, $\frac{-b^2}{4a} + c \ge 0$, $c \ge \frac{b^2}{4a}$, $4ac \ge b^2$.

Therefore, $|\langle \vec{u}. \vec{v} \rangle| \ge \|\vec{u}\| \cdot \|\vec{v}\|$.

Part II: (see [8]): let u and v are linearly dependent, such that $\vec{u} = \alpha\vec{v}$,then

$$\langle \vec{u} . \vec{v} \rangle = \langle \alpha\vec{u}. \vec{v} \rangle$$
$$= |\alpha|\|\vec{v}\|^2$$
$$= |\alpha|\|v\| \cdot \|v\|$$
$$= \|\alpha\vec{v}\| \cdot \|\vec{v}\|$$
$$= \|\vec{u}\| \cdot \|\vec{v}\|$$

Proof 2.4: by using the discriminate of the quadratic function: (see [4])

$$p(t) = \|\vec{u} + t\vec{v}\|^2$$
$$= \langle \vec{u} + t\vec{v}.\vec{u} + t\vec{v}\rangle$$
$$= \langle \vec{u}.\vec{u}\rangle + \langle \vec{u}.t\vec{v}\rangle + \langle t\vec{v}.\vec{u}\rangle + \langle t\vec{v}.t\vec{v}\rangle$$
$$= \langle \vec{u}.\vec{u}\rangle + t\langle \vec{u}.\vec{v}\rangle + t\langle \vec{v}.\vec{u}\rangle + \langle t\vec{v}.t\vec{v}\rangle$$
$$= \langle \vec{u}.\vec{u}\rangle + 2t\langle \vec{u}.\vec{v}\rangle + t^2\langle \vec{v}.\vec{v}\rangle$$
$$= t^2\langle \vec{v}.\vec{v}\rangle + 2t\langle \vec{u}.\vec{v}\rangle + \langle \vec{u}.\vec{u}\rangle$$
$$\langle \vec{v}.\vec{v}\rangle t^2 + 2\langle \vec{u}.\vec{v}\rangle t + \langle \vec{u}.\vec{u}\rangle \ \geq 0$$

Let $\langle \vec{v}.\vec{v}\rangle = a$, $\langle \vec{u}.\vec{v}\rangle = b$, $\langle \vec{u}.\vec{u}\rangle = c$, by substituting, we get: $at^2 + 2\,bt + c \geq 0$

This means that, this equation has no distinct two-real roots, thus the discriminate

$\Delta = 4b^2 - 4ac \leq 0$, then, $4b^2 \leq 4ac$

$$|\langle \vec{u}.\vec{v}\rangle|^2 \leq 4\langle \vec{v}.\vec{v}\rangle \cdot \langle \vec{u}.\vec{u}\rangle$$

Therefore, $|\langle \vec{u}.\vec{v}\rangle| \leq \ \|\vec{u}\| \cdot \|\vec{v}\|$.

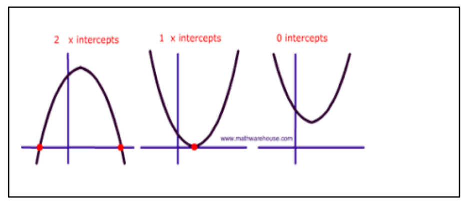

Figure: denote the of zeros to the quadratic function

Proof 2.5: by using determinant: (see [9])

For any scalar α,we use the nonnegative of the square $\|\vec{u} - \alpha\vec{v}\|^2$,such that:

$$0 \leq \|\vec{u} - \alpha\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + \alpha^2\|\vec{v}\|^2 - \langle \vec{u}.\alpha\vec{v}\rangle - \langle \alpha\vec{v}.\vec{u}\rangle$$
$$= (1 \quad \alpha) \cdot \begin{pmatrix} \|\vec{u}\|^2 & \langle \vec{u}.\vec{v}\rangle \\ \langle \vec{v}.\vec{u}\rangle & \|\vec{v}\|^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

Let $A = \begin{pmatrix} \|\vec{u}\|^2 & \langle \vec{u}.\vec{v}\rangle \\ \langle \vec{v}.\vec{u}\rangle & \|\vec{v}\|^2 \end{pmatrix}$ " A Hermitian". The above equation shows that 2 by1 vector $z = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ has non-zero first component, then $Z^* \cdot A \cdot Z \geq 0$, This non – negativity also holds when Z has zero first component as far $z = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$, we have:

$$Z^* \cdot A \cdot Z = \ |\alpha|^2\|\vec{v}\| \geq 0.$$

We conclude that A is positive semi defined, so it has non – negative determinant

$$0 \leq \det A \ = \|\vec{u}\|^2\|\vec{v}\|^2 - |\langle \vec{u}.\vec{v}\rangle|^2$$

From which the identity follows.

Proof 2.6: by using scalar: (see [10])

If either $\vec{u} = 0$ or $\vec{v} = 0$, then $\vec{u} \cdot \vec{v} = 0$ and $\|\vec{u}\| \cdot \|\vec{v}\| = 0$,so the equality hold. For the remainder of the proof, we will assume that $\vec{u}$ and $\vec{v}$ are non– zero vectors.

Let α and β be arbitrary scalars, then$\|\alpha\vec{u} + \beta\vec{v}\|^2 \geq 0$,using properties of length dot product, then we have

$$\|\alpha\vec{u}+\beta\vec{v}\|^2 = \langle \alpha\vec{u}+\beta\vec{v} . \alpha\vec{u}+\beta\vec{v}\rangle$$
$$= \alpha^2\langle\vec{u}.\vec{u}\rangle + \alpha\beta\langle\vec{u}.\vec{v}\rangle + \beta\alpha\langle\vec{v}.\vec{u}\rangle + \beta^2\langle\vec{v}.\vec{v}\rangle$$
$$= \alpha^2\|\vec{u}\|^2 + 2\alpha\beta\langle\vec{u}.\vec{v}\rangle + \beta^2\|\vec{v}\|^2.$$

Since this holds for all scalars α and β, we are free to choose $\alpha = \|\vec{v}\|$ and $\beta = \mp\|\vec{u}\|$, so

$$\|\alpha\vec{u}+\beta\vec{v}\|^2 = \|\vec{v}\|^2\|\vec{u}\|^2 + 2\|\vec{v}\|(\mp\|\vec{u}\|)\langle \mathrm{u}.\vec{v}\rangle + (\mp\|\vec{u}\|)^2\langle\vec{v}.\vec{v}\rangle$$
$$= 2\|\vec{v}\|^2\|\vec{u}\|^2 + 2\|\vec{v}\|(\mp\|\vec{u}\|)\langle\vec{u}.\vec{v}\rangle$$
$$= 2\|\vec{v}\|\|\vec{u}\|(\|\vec{v}\|\|\vec{u}\| \mp \langle\vec{u}.\vec{v}\rangle).$$

Since $\vec{u}$ and $\vec{v}$ are non – zero vector, $\|\vec{u}\| > 0$ and $\|\vec{v}\| > 0$, so $\|\alpha\vec{u}+\beta\vec{v}\|^2 > 0$, is true exactly when $(\|\vec{v}\|\|\vec{u}\| \mp \langle\vec{u}.\vec{v}\rangle) \geq 0$, that exactly when: $\|\vec{v}\|\|\vec{u}\| \geq \mp\langle\vec{u}.\vec{v}\rangle$. which is the same as $\|\vec{v}\|\|\vec{u}\| \geq |\langle\vec{u}.\vec{v}\rangle|$.

Note that $\|\vec{v}\|\|\vec{u}\| = |\langle\vec{u}.\vec{v}\rangle|$ exactly when $\|\alpha\vec{u}+\beta\vec{v}\|^2 = 0$,which is exactly when $\alpha\vec{u}+\beta\vec{v} = 0$,since u and $\vec{v}$ are non-zero vectors $\alpha\vec{u}+\beta\vec{v} = 0$,implies that $\alpha = \beta = 0$,or else $\vec{u}$ and $\vec{v}$ are scalar multiple of each other. Since α and β are non - zero. one vector must be scalar multiple of the others.

Proof 2.7: by using unit vectors: (see [10])

If either $\vec{u} = 0$ or $\vec{v} = 0$, ,then $\langle\vec{u}.\vec{v}\rangle = 0$ and $\|\vec{u}\| \cdot \|\vec{v}\| = 0$, so equality holds. For the remainder of the proof, we will assume that $\vec{u}$ and $\vec{v}$ are non – zero vectors. First suppose that x and y are unit vectors, using properties of length dot product, we have:

$$\|\mathrm{x} \pm \mathrm{y}\|^2 = \langle \mathrm{x} \pm \mathrm{y} . \mathrm{x} \pm \mathrm{y}\rangle$$
$$= \langle \mathrm{x}.\mathrm{x}\rangle \pm \langle \mathrm{x}.\mathrm{y}\rangle \pm \langle \mathrm{y}.\mathrm{x}\rangle + \langle \mathrm{y}.\mathrm{y}\rangle$$
$$= \|\mathrm{x}\|^2 \pm 2\langle \mathrm{x}.\mathrm{y}\rangle + \|\mathrm{y}\|^2.$$
$$= 1 \pm 2\langle \mathrm{x}.\mathrm{y}\rangle + 1.$$
$$= 2(1 \pm \langle \mathrm{x}.\mathrm{y}\rangle).$$

Since $\|\mathrm{x} \pm \mathrm{y}\|^2 \geq 0$, $1 \pm \langle \mathrm{x}.\mathrm{y}\rangle \geq 0$, this is the same as $1 \geq \pm\langle \mathrm{x}.\mathrm{y}\rangle$, thus $|\langle \mathrm{x}.\mathrm{y}\rangle| \leq 1$, further equality holds exactly when $\mathrm{x} \pm \mathrm{y} = 0$, which mean that $\mathrm{y} = \pm\mathrm{x}$. Now suppose that $\vec{u}$ and $\vec{v}$ are general non-zero vectors, then $\|\vec{u}\| \geq 0$ and $\|\vec{v}\| \geq 0$, so $\mathrm{x} = \frac{1}{\|\vec{u}\|}\vec{u}$ and $\mathrm{y} = \frac{1}{\|\vec{v}\|}\vec{v}$ are unit vectors. Then

$$\left|\langle \frac{1}{\|\vec{u}\|}\vec{u} . \frac{1}{\|\vec{v}\|}\vec{v}\rangle\right| \leq 1$$
$$\frac{1}{\|\vec{u}\|} \cdot \frac{1}{\|\vec{v}\|} |\langle\vec{u} . \vec{v}\rangle| \leq 1$$

Therefore, $|\langle\vec{u} . \vec{v}\rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$.

Equality holds exactly when $\frac{1}{\|\vec{u}\|}\vec{u} = \pm\frac{1}{\|\vec{v}\|}\vec{v}$,which means exactly when one vector is a scalar multiple of the other.

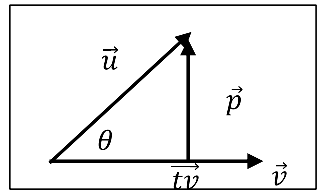

Figure 2denote the cosine formula

Proof 2.8: by using cosine formula: (see [4])

Let u and v be any two vectors, for scalar product we have

$\langle \vec{u}\,.\,\vec{v}\rangle = |\vec{u}|\cdot|\vec{v}|\cos\theta$

$\langle \vec{u}\,.\,\vec{v}\rangle = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$

$$\frac{|\langle \vec{u}\,.\vec{v}\rangle|}{\|\vec{u}\|\cdot\|\vec{v}\|} = |\cos\theta|\ ,$$

We have, $-1 \le \cos\theta \le 1$, $|\cos\theta| \le 1$ Then, $\frac{|\langle \vec{u}\,.\vec{v}\rangle|}{\|\vec{u}\|\cdot\|\vec{v}\|} = 1$

Therefore, $|\langle \vec{u}\,.\,\vec{v}\rangle| \le \|\vec{u}\|\cdot\|\vec{v}\|$

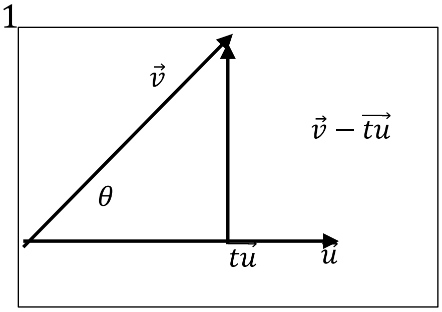

Figure 3 : cosine definition

Proof 2.9: by using cosine definition: (see [8])

From Pythagorean theorem, we have:

$$\|\vec{v}\|^2 = \left\|\overrightarrow{tu}\right\|^2 + \left\|\vec{v} - \overrightarrow{tu}\right\|^2.$$

$$\langle \vec{v}\,.\,\vec{v}\rangle = \mathrm{t}^2\langle \vec{u}\,.\,\vec{u}\rangle + \langle \vec{v} - \overrightarrow{tu}.\vec{v} - \overrightarrow{tu}\rangle.$$

$$\langle \vec{v}\,.\,\vec{v}\rangle = \mathrm{t}^2\langle \vec{u}\,.\,\vec{u}\rangle + \langle \vec{v}\,.\,\vec{v}\rangle - 2\mathrm{t}\langle \vec{u}\,.\,\vec{v}\rangle + \mathrm{t}^2\langle \vec{u}\,.\,\vec{u}\rangle.$$

Cancelling $\langle \vec{v}\,.\,\vec{v}\rangle$ and collecting terms yields: $\mathrm{t} = \frac{|\langle \vec{u}\,.\vec{v}\rangle|}{\|\vec{u}\|\cdot\|\vec{u}\|}$, By the assumption that θ lies strictly between 0 and 90 degrees, t is positive. By the definition of cosine, we have

$$\cos\theta = \frac{\|\overrightarrow{tu}\|}{\|\vec{v}\|} = \frac{\langle \vec{u}\,.\vec{v}\rangle}{\langle \vec{u}\,.\vec{u}\rangle}\cdot\frac{\langle \vec{u}\,.\vec{u}\rangle^{\frac{1}{2}}}{\|\vec{v}\|} = \frac{\langle \vec{u}\,.\vec{v}\rangle}{\|\vec{u}\|\cdot\|\vec{v}\|}.$$

Rewriting, $\langle \vec{u}\,.\,\vec{v}\rangle = \|\vec{u}\|\|\vec{v}\|\cos\theta$. If $\|\vec{u}\| = \|\vec{v}\| = 1$, then $\cos\theta = \langle \vec{u}\,.\,\vec{v}\rangle$. This implies the Cauchy - Schwarz inequality, for θ strictly between 0 and 90 degrees, $0 < \cos\theta < 1$, hence $0 < \langle \vec{u}\,.\,\vec{v}\rangle < \|\vec{u}\|\|\vec{v}\|$. In particular, $\langle \vec{u}\,.\,\vec{v}\rangle = 0$ if and only if $\vec{u}$ and $\vec{v}$ are orthogonal.

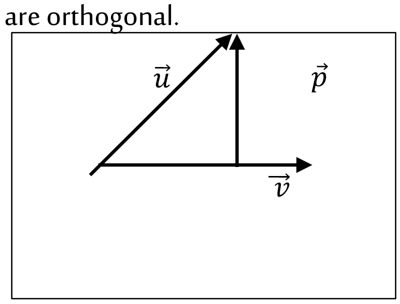

Figure 4: orthogonal decomposition

Proof 2.10: by using orthogonal decomposition: (see [2]).

if $\vec{v} = 0$, then both sides of the inequality are zero. Hence assume that $\vec{v} \neq 0$.Consider the orthogonal decomposition:

$$\vec{u} = \vec{p} + \frac{\langle \vec{u}\,.\vec{v}\rangle}{\langle \vec{v}\,.\vec{v}\rangle}\cdot\vec{v}$$

$$\langle \vec{u}\,.\,\vec{u}\rangle = \langle\ \vec{p} + \frac{\langle \vec{u}\,.\vec{v}\rangle}{\langle \vec{v}\,.\vec{v}\rangle}\cdot\vec{v}.\vec{p} + \frac{\langle \vec{u}\,.\vec{v}\rangle}{\langle \vec{v}\,.\vec{v}\rangle}\cdot\vec{v}\rangle.$$

$$\|\vec{u}\,\|^2 = \langle \vec{p}.\vec{p}\rangle + \langle \vec{p}.\frac{\langle \vec{u}\,.\,\vec{v}\rangle}{\langle \vec{v}\,.\,\vec{v}\rangle}\cdot\vec{v}\rangle + \langle\frac{\langle \vec{u}\,.\,\vec{v}\rangle}{\langle \vec{v}\,.\,\vec{v}\rangle}\cdot\vec{v}.\vec{p}\rangle + \langle\frac{\langle \vec{u}\,.\,\vec{v}\rangle}{\langle \vec{v}\,.\,\vec{v}\rangle}\cdot\vec{v}.\frac{\langle \vec{u}}{\langle \vec{v}\ ..}\ ,$$

$$\|\vec{u}\|^2 = \langle \vec{p}.\vec{p}\rangle + \frac{\langle \vec{u}\,.\,\vec{v}\rangle^2}{\langle \vec{v}\,.\,\vec{v}\rangle^2}\cdot\langle \vec{v}\,.\,\vec{v}\rangle$$

$$\|\vec{u}\|^2 = \|\vec{p}\|^2 + \frac{\langle \vec{u}\,.\vec{v}\rangle^2}{\langle \vec{v}\,.\vec{v}\rangle}.$$

$$\|\vec{u}\|^2 \geq \frac{\langle \vec{u}\,.\,\vec{v}\rangle^2}{\langle \vec{v}\,.\,\vec{v}\rangle}$$

$\|\vec{u}\|^2\|\vec{v}\|^2 \geq \langle \vec{u}\,.\,\vec{v}\rangle^2$, hence $\langle \vec{u}\,.\,\vec{v}\rangle < \|\vec{u}\|\cdot\|\vec{v}\|$.

Proof 2.11: by using mathematical induction: (see [4]).

$$|\mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2 + \mathrm{u}_3\mathrm{v}_3 + \cdots .\,\mathrm{u}_n\mathrm{v}_n|^2 \leq (\mathrm{u}_1 + \mathrm{u}_2 + \mathrm{u}_3 + \cdots + \mathrm{u}_n)^2(\mathrm{v}_1 + \mathrm{v}_2 + \mathrm{v}_3 + \cdots + \mathrm{v}_n)^2$$

Beginning the induction at $n = 1$, the case is trivial, let $n = 2$, then we have:

$$(\mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2)^2 = {\mathrm{u}_1}^2{\mathrm{v}_1}^2 + 2\mathrm{u}_1\mathrm{u}_2\mathrm{v}_1\mathrm{v}_2 + {\mathrm{u}_2}^2{\mathrm{v}_2}^2.$$

$$\leq {\mathrm{u}_1}^2{\mathrm{v}_1}^2 + {\mathrm{u}_1}^2{\mathrm{v}_2}^2 + {\mathrm{u}_2}^2{\mathrm{v}_1}^2 + {\mathrm{u}_2}^2{\mathrm{v}_2}^2$$

$$= ({\mathrm{u}_1}^2 + {\mathrm{u}_2}^2)\cdot({\mathrm{v}_1}^2 + {\mathrm{v}_2}^2).$$

Which implies that the inequality holds for $n = 2$. Assume that the inequality holds for an arbitrary integer k, i.e.,

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} {a_i}^2\right)\left(\sum_{i=1}^{k} {b_i}^2\right)$$

Using the induction hypothesis, one has

$$\sqrt{\sum_{i=1}^{k+1} {a_i}^2}\cdot\sqrt{\sum_{i=1}^{k+1} {b_i}^2} = \sqrt{\sum_{i=1}^{k} {a_i}^2 + a_{k+1}^2}\cdot\sqrt{\sum_{i=1}^{k} {b_i}^2 + b_{k+1}^2}$$

$$\geq \sqrt{\sum_{i=1}^{k} {a_i}^2}\cdot\sqrt{\sum_{i=1}^{k} {b_i}^2} + |a_{k+1}b_{k+1}|$$

$$\geq \sum_{i=1}^{k}|a_i b_i| + |a_{k+1}b_{k+1}| = \sum_{i=1}^{k+1}|a_i b_i|.$$

It means that the inequality holds for $n = k + 1$,we thus conclude that the inequality holds for all-natural numbers. This complete the proof of the inequality.

Corollary 2.1: If $\langle\cdot.\cdot\rangle$ is an inner product on a vector space X ,then for all $\vec{u} = (\mathrm{u}_1.\mathrm{u}_2)$, $\vec{v} = (\mathrm{v}_1.\mathrm{v}_2)$, then we have**:**

$$(\mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2)^2 \leq (\mathrm{u}_1^2 + \mathrm{u}_2^2)(\mathrm{v}_1^2 + \mathrm{v}_2^2), \forall\, \mathrm{u}.\mathrm{v} \in \mathcal{R}^2.$$

Proof: Let $(\mathrm{u}_1\mathrm{v}_2 - \mathrm{u}_2\mathrm{v}_1)$ be a quantity, then by squaring it, we have:

$$(\mathrm{u}_1\mathrm{v}_2 - \mathrm{u}_2\mathrm{v}_1)^2 \geq 0$$

$${\mathrm{u}_1}^2{\mathrm{v}_2}^2 - 2\mathrm{u}_1\mathrm{u}_2\mathrm{v}_1\mathrm{v}_2 + {\mathrm{u}_2}^2{\mathrm{v}_1}^2 \geq 0$$

$${\mathrm{u}_1}^2{\mathrm{v}_2}^2 + {\mathrm{u}_2}^2{\mathrm{v}_1}^2 \geq 2\mathrm{u}_1\mathrm{u}_2\mathrm{v}_1\mathrm{v}_2$$

By adding ${\mathrm{u}_1}^2{\mathrm{v}_1}^2 + {\mathrm{u}_2}^2{\mathrm{v}_2}^2$ to both sides, we get:

$$u_1{}^2v_2{}^2 + u_2{}^2v_1{}^2 + u_1{}^2v_1{}^2 + u_2{}^2v_2{}^2 \ \geq\ 2u_1u_2v_1v_2 + u_1{}^2v_1{}^2 + u_2{}^2v_2{}^2$$
$$u_1{}^2(v_1{}^2 + v_2{}^2) + u_2{}^2(v_1{}^2 + v_2{}^2) \geq u_1{}^2v_1{}^2 + 2u_1u_2v_1v_2 + u_2{}^2v_2{}^2$$
$$(u_1{}^2 + u_2{}^2)(v_1{}^2 + v_2{}^2) \geq (u_1u_2 + v_1v_2)^2$$
$$(u_1u_2 + v_1v_2)^2 \ \leq\ (u_1{}^2 + u_2{}^2)\cdot(v_1{}^2 + v_2{}^2).$$

corollary 2.2: If $\langle\cdot,\cdot\rangle$ is an inner product on a vector space X, then for all $\vec{u} = (u_1.u_2.u_3)$, $\vec{v} = (v_1.v_2.v_3)$, then we have:

$$(u_1v_1 + u_2v_2 + u_3v_3)^2 \leq (u_1^2 + u_2^2 + u_3^2)\cdot(v_1^2 + v_2^2 + v_3^2),\ \forall\, u.v \in \mathcal{R}^3.$$

Proof:

Assume we have the following positive quantity:

$$(u_1v_2 - u_2v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_2v_3 - u_3v_2)^2 \geq 0$$
$$u_1{}^2v_2{}^2 - 2u_1u_2v_1v_2 + u_2{}^2v_1{}^2 + u_1{}^2v_3{}^2 - 2u_1u_3v_1v_3 + u_3{}^2v_1{}^2 + u_2{}^2v_3{}^2 - 2u_2u_3v_2v_3 + u_3{}^2v_2{}^2 \geq 0.$$

By adding $u_1{}^2v_1{}^2 + u_2{}^2v_2{}^2 + u_3{}^2v_3{}^2$ to both sides, we get:

$$u_1{}^2v_2{}^2 - 2u_1u_2v_1v_2 + u_2{}^2v_1{}^2 + u_1{}^2v_3{}^2 - 2u_1u_3v_1v_3 + u_3{}^2v_1{}^2 + u_2{}^2v_3{}^2 - 2u_2u_3v_2v_3 + u_3{}^2v_2{}^2 + u_1{}^2v_1{}^2 + u_2{}^2v_2{}^2 + u_3{}^2v_3{}^2 \geq u_1{}^2v_1{}^2 + u_2{}^2v_2{}^2 + u_3{}^2v_3{}^2.$$

$$(u_1{}^2v_1{}^2 + u_1{}^2v_2{}^2 + u_1{}^2v_3{}^2) + (u_2{}^2v_1{}^2 + u_2{}^2v_2{}^2 + u_2{}^2v_3{}^2) + u_3{}^2v_1{}^2 + u_3{}^2v_2{}^2 + u_3{}^2v_3{}^2 \geq (u_1u_2v_1v_2 + u_1{}^2v_1{}^2 + u_1u_2v_1v_2) + u_1u_3v_1v_3 + u_2{}^2v_2{}^2 + u_1u_3v_1v_3 + (u_2u_3v_2v_3 + u_3{}^2v_3{}^2 + u_2u_3v_2v_3).$$

$$u_1{}^2(v_1{}^2 + v_2{}^2 + v_3{}^2) + u_2{}^2(v_1{}^2 + v_2{}^2 + v_3{}^2) + u_3{}^2(v_1{}^2 + v_2{}^2 + v_3{}^2) \geq u_1v_1(u_1v_1 + u_2v_2 + u_3v_3) + u_2v_2(u_1v_1 + u_2v_2 + u_3v_3) + u_3v_3(u_1v_1 + u_2v_2 + u_3v_3).$$

$$(u_1{}^2 + u_2{}^2 + u_3{}^2)(v_1{}^2 + v_2{}^2 + v_3{}^2) \geq (u_1v_1 + u_2v_2 + u_3v_3)(u_1v_1 + u_2v_2 + u_3v_3)$$
$$(u_1{}^2 + u_2{}^2 + u_3{}^2)(v_1{}^2 + v_2{}^2 + v_3{}^2) \geq (u_1v_1 + u_2v_2 + u_3v_3)^2$$
$$(u_1v_1 + u_2v_2 + u_3v_3)^2 \leq (u_1{}^2 + u_2{}^2 + u_3{}^2)(v_1{}^2 + v_2{}^2 + v_3{}^2)$$

Hence the inequality holds.

method(1) : using Pythagorean theorem
method(2) : using the inner product
method(3) : using quadratic function
method(4) : using discriminant of the quadratic equation
Proof method(5) : using using determinant
method(6) : using using scalars
method(7) : using unit vectors
method(8) : using cosine formula
method(9) : using orthogonal
method(10) : using orthogonal decomposition
method(11) : using mathematical induction

Table -Cauchy-schwarz Inequality – methods of proof

3- Applications:

The Cauchy – Schwarz inequality is an important inequality, that it has many applications as follows:

Theorem3.1: Triangle Inequality for inner product spaces: (see [2]).

For all $\vec{u}$ and $\vec{v} \in \mathrm{V}$, we have: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\|+\|\vec{v}\|$.

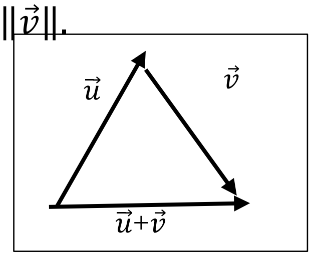

Figure 5: triangle inequality

Proof:

by straight forward calculation, we obtain:

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v}.\vec{u}+\vec{v}\rangle \\
&= \langle \vec{u}.\vec{u}\rangle + \langle \vec{u}.\vec{v}\rangle + \langle \vec{v}.\vec{u}\rangle + \langle \vec{v}.\vec{v}\rangle. \\
&= \|\vec{u}\|^2 + \langle \vec{u}.\vec{v}\rangle + \overline{\langle \vec{u}.\vec{v}\rangle} + \|v\|^2. \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2Re\langle \vec{u}.\vec{v}\rangle.
\end{aligned}$$

Note that: $Re\langle \vec{u}.\vec{v}\rangle \leq |\langle \vec{u}.\vec{v}\rangle|$, so that by using Cauchy - Schwarz inequality we obtain:

$$\|\vec{u}+\vec{v}\|^2 \leq \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle \vec{u}.\vec{v}\rangle$$

$$\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2.$$

Taking the square root of both sides, gives the triangle inequality.

Here we have some inequalities generated by the Cauchy- Schwarz inequality:

Inequality 3.2: (see [2]).

Let a, b, c be any real numbers greater than one, such that: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=3$, then we have: $\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1} \le \sqrt{a+b+c+d}$.

Proof:

Let, $\vec{x} = \left(\frac{\sqrt{a-1}}{\sqrt{a}}.\frac{\sqrt{b-1}}{\sqrt{b}}.\frac{\sqrt{c-1}}{\sqrt{c}}.\frac{\sqrt{d-1}}{\sqrt{d}}\right)$, $\vec{y} = \left(\frac{1}{\sqrt{bcd}}.\frac{1}{\sqrt{acd}}.\frac{1}{\sqrt{abd}}.\frac{1}{\sqrt{abc}}\right)$, By applying the Cauchy Schwarz inequality $\langle \vec{x}.\vec{y}\rangle < \|\vec{x}\|\|\vec{y}\|$, this yield to :

$$\frac{\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1}}{\sqrt{abcd}} \le \sqrt{\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}+\frac{d-1}{d}} \cdot \sqrt{\frac{1}{bcd}+\frac{1}{acd}+\frac{1}{abd}+\frac{1}{abc}},$$

Now multiply $\sqrt{abcd}$ across the sign, we get:

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1} \le \sqrt{\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}+\frac{d-1}{d}} \cdot \sqrt{a+b+c+d}$$

$$\le \sqrt{1-\frac{1}{a}+1-\frac{1}{b}+1-\frac{1}{c}+1-\frac{1}{d}} \cdot \sqrt{a+b+c}$$

$$\le \sqrt{4-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)} \cdot \sqrt{a+b+c+d}$$

$$\le \sqrt{4-3} \cdot \sqrt{a+b+c+d}$$

$$\le \sqrt{1} \cdot \sqrt{a+b+c+d}$$

$$\therefore \sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}+\sqrt{d-1} \le \sqrt{a+b+c+d}.$$

corollary 3.1:

Let $a_1, a_2, a_3,\ldots,a_n \in (1.\infty)$, such that: $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\ldots\ldots.\frac{1}{a_n}=n-1$, then we have:

$$\sqrt{a_1-1}+\sqrt{a_2-1}+\sqrt{a_3-1}+\cdots+\sqrt{a_n-1} \le \sqrt{a_1+a_2+a_3+\cdots+a_n}.$$

Proof:

Let, $$\vec{x} = \left(\frac{\sqrt{a_1-1}}{a_1}.\frac{\sqrt{a_2-1}}{a_2}.\frac{\sqrt{a_3-1}}{a_3}\ldots.\frac{\sqrt{a_n-1}}{a_n}\right).$$

$$\vec{y} = \left(\frac{1}{\sqrt{a_2a_3\ldots a_n}}.\frac{1}{\sqrt{a_1a_3\ldots a_n}}.\frac{1}{\sqrt{a_1a_2a_4\ldots a_n}}\ldots.\frac{1}{\sqrt{a_1a_2a_4\ldots a_{n-1}}}\right),$$

By applying the Cauchy Schwarz inequality, $\langle \mathrm{x}.\mathrm{y}\rangle < \|\mathrm{x}\|\|\mathrm{y}\|$, we get:

$$\frac{\sqrt{a_1-1}}{\sqrt{a_1a_2a_3\ldots a_n}}+\frac{\sqrt{a_2-1}}{\sqrt{a_1a_2a_3\ldots a_n}}+\frac{\sqrt{a_3-1}}{\sqrt{a_1a_2a_3\ldots a_n}}+\cdots \leq \sqrt{\frac{a_1-1}{a_1}+\frac{a_2-1}{a_2}+\frac{a_3-1}{a_3}+\cdots}\cdot$$
$$\sqrt{\frac{1}{a_2a_3\ldots a_n}+\frac{1}{a_1a_3\ldots a_n}+\frac{1}{a_1a_2\ \ldots a_n}+\cdots},$$

Now multiply $\sqrt{a_1a_2a_3\ \ldots\ a_n}$ across the sign, we get:

$$\sqrt{a_1-1}+\sqrt{a_2-1}+\sqrt{a_3-1}+\cdots \leq \left(1-\frac{1}{a_1}+1-\frac{1}{a_2}+1-\frac{1}{a_3}+\cdots+1-\frac{1}{a_n}\right)\cdot$$
$$\sqrt{a_{1+}a_2+a_3+\cdots+a_n}$$
$$\leq \sqrt{n-\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right)}\cdot\sqrt{a_{1+}a_2+a_3+\cdots+a_n}$$
$$\leq \sqrt{n-(n-1)}\cdot\sqrt{a_{1+}a_2+a_3+\cdots+a_n}$$
$$\leq \sqrt{1}\cdot\sqrt{a_{1+}a_2+a_3+\cdots+a_n}.$$
$$\therefore \sqrt{a_1-1}+\sqrt{a_2-1}+\sqrt{a_3-1}+\cdots+\sqrt{a_n-1} \leq \sqrt{a_1+\ a_2+\ a_3+\cdots+a_n}\,.$$

Inequality 3.2:

if $a.b.c.d$ belong to $\mathcal{R}$, then $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$

Proof:

Let, $\vec{x} = (a.b.c.d).\ \overrightarrow{y} = (1.1.1.1)$, be any two vectors, by applying the Cauchy Schwarz inequality, $\langle\vec{x}.\overrightarrow{y}\rangle < \|\vec{x}\|\|\overrightarrow{y}\|$, we get:

$$a+b+c+d \leq \sqrt{a^2+b^2+c^2+d^2}\sqrt{1^2+1^2+1^2+1^2}$$
$$\leq \sqrt{a^2+b^2+c^2+d^2}\sqrt{4},$$

squaring both sides, then we have:

$$(a+b+c+d)^2 \leq\ 4(a^2+b^2+c^2).$$

corollary 3.4:

if $a_1, a_2, a_3,\ldots,a_n$ belong to $\mathcal{R}$, then:

$$(a_1+\ a_2+\ a_3+\cdots+a_n)^2 \leq n(a_1^2+a_2^2+a_3^2+\cdots+a_n^2)$$

Proof:

Let, $\vec{x} = (a_1.a_2.a_3.\ldots.a_n).\ \overrightarrow{y} = (k.k.k\ \ldots.k)$, By applying the Cauchy Schwarz inequality,$\langle\vec{x}.\overrightarrow{y}\rangle < \|\vec{x}\|\|\overrightarrow{y}\|$, we get:

$$a_1k + a_2\mathrm{k} + a_3\mathrm{k} + \cdots + \mathrm{k}a_n \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}\sqrt{k^2 + k^2 + k^2 + \cdots k^2}$$

$$\mathrm{k}(a_1 + a_2 + a_3 + \cdots + a_n) \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot \sqrt{nk^2}$$

$$\mathrm{k}(a_1 + a_2 + a_3 + \cdots + a_n) \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot k\sqrt{n}$$

$$(a_1 + a_2 + a_3 + \cdots + a_n) \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot \sqrt{n}$$

Squaring both sides, then we get:

$$(a_1 + a_2 + a_3 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2).$$

Corollary3.5: given $a_1, a_2, a_3,\ldots,a_n$, such that: $a_1 + a_2 + a_3 + \cdots + a_n = 1$, then the minimum value of $a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2$, is $\frac{1}{n}$.

Proof:

Let, $\vec{x} = (a_1.a_2.a_3.\ldots.a_n)$, $\vec{y} = (1.1.1\ldots.1)$, be any two vectors, By applying the Cauchy Schwarz inequality, $\langle\vec{x}.\vec{y}\rangle < \|\vec{x}\|\|\vec{y}\|$, we get: $a_1 + a_2 + a_3 + \cdots + a_n \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}\sqrt{1^2 + 1^2 + 1^2 + \cdots + 1^2}$

$$(a_1 + a_2 + a_3 + \cdots + a_n) \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot \sqrt{n1^2}$$

$$(a_1 + a_2 + a_3 + \cdots + a_n) \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot \sqrt{n}$$

$$1 \le \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot \sqrt{n}$$

squaring both sides, then we get:

$$(1)^2 \le n(a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2).$$

$$\frac{1}{n} \le a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2$$

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 \le \frac{1}{n}.$$

Therefore, the minimum value of $a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2$, is $\frac{1}{n}$.

Corollary 3.6: if $a.b.c$ belong to $\mathcal{R}^+$, then the minimum value of $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ is equal 9.

Proof:

Let: $\vec{x} = (\sqrt{a}.\sqrt{b}.\sqrt{c})$. $\vec{y} = \left(\frac{1}{\sqrt{a}}.\frac{1}{\sqrt{b}}.\frac{1}{\sqrt{c}}\right)$, be any two vectors, by applying the Cauchy Schwarz inequality, we get:

$$1+1+1 \leq \sqrt{a+b+c} \cdot \sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

$$3 \leq \sqrt{a+b+c} \cdot \sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

squaring both sides, then we get: $9 \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$\therefore (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$

$\therefore$ the minimum value of $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is equal 9.

Corollary 3.7: if $a_1, a_2, a_3, \ldots, a_n$ belong to $\mathcal{R}$, then

$$(a_1+\ a_2+\ a_3+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right) \geq n^2..$$

Proof:

Let, $\vec{x} = \left(\sqrt{a_1}.\sqrt{a_2}.\sqrt{a_3} \ldots .\sqrt{a_n}\right).\ \vec{y} = \left(\frac{1}{\sqrt{a_1}}.\frac{1}{\sqrt{a_2}}.\frac{1}{\sqrt{a_3}} \ldots .\frac{1}{\sqrt{a_n}}\right)$, be any two vectors,

by applying the Cauchy Schwarz inequality, we get:

$$1+1+1+\cdots+1 \leq (a_1+\ a_2+\ a_3+\cdots+a_n) \cdot \sqrt{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$$

$$n \leq \sqrt{a_1+\ a_2+\ a_3+\cdots+a_n} \cdot \sqrt{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$$

squaring both sides, then we get: $n^2 \leq (a+b+c)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right)$

$$\therefore (a_1+\ a_2+\ a_3+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right) \geq n^2 .$$

Corollary 3.8: let $a_1, a_2, \ldots, a_n$, belong to $\mathcal{R}$, such that: $a_1+\ a_2+a_3+\cdots+a_n = 1$, then the minimum value of: $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}$, is n^2.

Proof:

Let, $\vec{x} = \left(\sqrt{a_1}.\sqrt{a_2}.\sqrt{a_3} \ldots .\sqrt{a_n}\right).\vec{y} = \left(\frac{1}{\sqrt{a_1}}.\frac{1}{\sqrt{a_2}}.\frac{1}{\sqrt{a_3}} \ldots .\frac{1}{\sqrt{a_n}}\right)$,be any two vectors, By applying the Cauchy Schwarz inequality, we get:

$$1+1+1+\cdots+1 \leq (a_1+\ a_2+\ a_3+\cdots+a_n) \cdot \sqrt{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$$

$$n \leq \sqrt{a_1+\ a_2+\ a_3+\cdots+a_n} \cdot \sqrt{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$$

squaring both sides, then we get:

$$n^2 \leq (a+b+c)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right)$$
$$\therefore (a_1+\ a_2+\ a_3+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right) \geq n^2.$$
$$\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right) \geq n^2$$

then the minimum value of $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}$, is n^2.

Corollary 3.9: If $x^2+y^2+z^2+w^2=1$, $x.y.z.w$ belong to $\mathcal{R}$, then:

$$x+2y+3z+4w \leq \sqrt{30}.$$

Proof:

$\vec{x} = (x.y.z.w)$. $\vec{y} = (1.2.3.4)$, be any two vectors, by applying the Cauchy Schwarz inequality, we get:

$$x+2y+3z+4w \leq \sqrt{x^2+y^2+z^2+w^2}\cdot\sqrt{1^2+2^2+3^2+4^2}$$
$$x+2y+3z+4w \leq \sqrt{30}.$$

Corollary 3.10:

If ${a_1}^2+\ {a_2}^2+\ {a_3}^2+\cdots+{a_n}^2=1$, $a_1, a_2, a_3, \ldots, a_n$ belong to $\mathcal{R}$, then

$$a_1+\ 2a_2+\ 3a_3+\cdots+\ na_n \leq \sqrt{\frac{n(n+1)(2n+1)}{6}}.$$

Proof:

$\vec{x} = (a_1.a_2.a_3.\ldots.a_n)$. $\vec{y} = (1.2.3.4\ldots\ldots.n)$, by applying the Cauchy Schwarz inequality, we get:

$$a_1+2a_2+3a_3+4a_4+\cdots+na_n \leq \sqrt{{a_1}^2+\ {a_2}^2+\ {a_3}^2+\cdots+{a_n}^2}\cdot\sqrt{1^2+2^2+3^2+4^2+\cdots+n^2}$$

$$a_1+2a_2+3a_3+4a_4+\cdots+na_n \leq \sqrt{1}\cdot\sqrt{1^2+2^2+3^2+4^2+\cdots+n^2}$$
$$a_1+2a_2+3a_3+4a_4+\cdots+na_n \leq \sqrt{1^2+2^2+3^2+4^2+\cdots+n^2}$$
$$a_1+2a_2+3a_3+4a_4+\cdots+na_n \leq \sqrt{\frac{n(n+1)(2n+1)}{6}}.$$

Problem 3.1: if $a.b.c$ belong to $\mathcal{R}$, then prove that: $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \leq 6$

Proof:

Let $\vec{x} = (\sqrt{a}.\sqrt{b}.\sqrt{c}).\ \ \vec{y} = \left(\frac{1}{\sqrt{a}}.\frac{1}{\sqrt{b}}.\frac{1}{\sqrt{c}}\right)$, by applying the Cauchy Schwarz inequality, $\langle \vec{x}.\vec{y}\rangle < \|\vec{x}\|\|\vec{y}\|$, we get: $1+1+1 \le \sqrt{a+b+c}\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$.

$$3 \le \sqrt{a+b+c}\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

squaring both sides, then we have:

$$9 \le (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

$$9 \le \frac{(a+b+c)}{a}+\frac{(a+b+c)}{b}+\frac{(a+b+c)}{c}.$$

$$9 \le \frac{a}{a}+\frac{b+c}{a}+\frac{b}{b}+\frac{a+c}{b}+\frac{c}{c}+\frac{a+b}{c}.$$

$$9 \le 1+\frac{b+c}{a}+1+\frac{a+c}{b}+1+\frac{a+b}{c}.$$

$$9 \le 3+\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}$$

$$6 \le \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}.$$

Problem 3.2:

if $a.\,b$ and $.\,c$ are the sides of a triangle, then: $\frac{\mathrm{a}}{\mathrm{b+c-a}}+\frac{\mathrm{b}}{\mathrm{c+a-c}}+\frac{\mathrm{c}}{\mathrm{a+b-c}} \ge 3$

Proof:

Let, $\vec{x} = (\sqrt{a}.\sqrt{b}.\sqrt{c}).\ \vec{y} = \left(\frac{1}{\sqrt{a}}.\frac{1}{\sqrt{b}}.\frac{1}{\sqrt{c}}\right)$, be any two vectors, by applying the Cauchy Schwarz inequality, $\langle \vec{x}.\vec{y}\rangle < \|\vec{x}\|\|\vec{y}\|$,

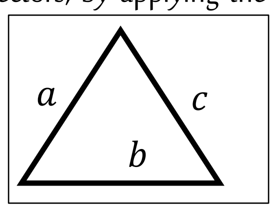

Figure 6 : triangle of sides

we get: $$1+1+1 \le \sqrt{a+b+c}\cdot\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

$$3 \le \sqrt{a+b+c}\cdot\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

squaring both sides, then we have: $9 \le (a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

$$9 \le \frac{(a+b+c)}{a}+\frac{(a+b+c)}{b}+\frac{(a+b+c)}{c}.$$

$$9 \le \frac{a}{a}+\frac{b+c}{a}+\frac{b}{b}+\frac{a+c}{b}+\frac{c}{c}+\frac{a+b}{c}.$$

$$9 \le 1+\frac{b+c}{a}+1+\frac{a+c}{b}+1+\frac{a+b}{c}.$$

$$9 \le 3+\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}$$

$$6-3 \le \frac{b+c}{a}-\frac{a}{a}+\frac{a+c}{b}-\frac{b}{b}+\frac{a+b}{c}-\frac{c}{c}.$$

$$3 \leq \frac{b+c-a}{a} + \frac{a+c-b}{b} + \frac{a+b-c}{c}$$
$$\therefore \quad \frac{b+c-a}{a} + \frac{a+c-b}{b} + \frac{a+b-c}{c} \geq 3.$$

DISCUSSION:

1- The study was able to achieve fourteen results in total are different inequalities at the sides of each one of them a unique series suggests the generation of more in many spaces, which can be called the space of Cauchy inequalities, which can contribute to the increase of knowledge stocks and the possibility of application in different scientific aspects.

2- The importance of this inequality of Cauchy Schwarz of vectors, in the multiple proofs, which may be 40 proofs, this paper was able to study 11 of them, which are based on the following ideas: Pythagorean theorem, inner product of vectors, quadratic function, cosine formula, orthogonality, mathematical induction.

3- These results enable us to find the minimum value of the series or the maximum value of it, which contributes to the concept of optimization.

4- Some of the results were directly related to the sides of the triangle, which contributes to the generation of new and diverse relationships in the geometric space.

4- **Conclusion:** Cauchy - Schwarz inequality differentiated by many different proofs, which gained importance in producing a variety of mathematical inequalities and providing proofs. This paper provided different proofs for this inequality that allowed to conduct such studies.

RECOMMENDATIONS:

Based on the results of our research and discussions, we recommend the following studies:

1. To conduct a study dealing with different proofs of this inequality in the space of functions with applications in the space of the numerical analysis.
2. A comparative study dealing with Cauchy Schwarz inequality with some of the famous inequalities with applications in the geometric spaces.
3. Conducting an advanced study on the role of Cauchy Schwarz inequality in solving mathematical problems with applications on topological space.

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متراجحة كوشي شوارتز للمتجهات في فضاء الضرب الداخلي مع التطبيقات

الملخص: تفردت متباينة كوشي شوارتز بالعديد من البراهين المختلفة، والتي لم يسمح المجال بذكرها جميعاً، مما أكسبها أهمية في توليد متباينات رياضية متنوعة وتقديم براهين لها. تناولت هذه الورقة براهين مختلفة لهذه النظرية في إطار فضاء الضرب الداخلي للمتجهات، مما أسهم في تأسيس متباينات رياضية مختلفة أبرزها التطبيقات التي تظهر علاقات متعددة على أضلاع المثلث .

الكلمات المفتاحية: متباينة كوشي شوارتز، فضاء الضرب الداخلي، معيار المتجه، الضرب القياسي، متجه الوحدة، تحليل المتجه، الاستقراء الرياضي