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On the Solution of Fractional Heat Diffusion

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Abstract: When the derivative of a function is non-integer order, e.g. the 1/2 derivative, known as fractional calculus. The fractional heat equation is a generalization of the standard heat equation as it uses an arbitrary derivative order close to 1 for the time derivative. We present a stander solution to an initial-boundary-value - heat equation problem and the solution to an initial-boundary-value - fractional heat equation problem. Our aim is to apply fractional Laplace trance form method and Fourier trance form method to solve the heat diffusion equations with fractional derivative and integral. In this study we used Fourier and Laplace transform methods. We conclude that the fractional heat equation is a physically legitimate generalization of the standard heat equation that might be used for values $\alpha \approx 1$. As expected all solutions sufficiently close to α satisfy the boundary conditions and display physically realistic properties

Keywords: Fractional Calculus; Heat Equation; Fourier and Laplace transform

1. Introduction

Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus derivative and integral operators [4]. Historically as most writers have mentioned the birth of so-called 'Fractional Calculus'. In a letter dated September 30th, 1695 L'Hopital [4] wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function f(x) = x, $\frac{D^n x}{Dx^n}$. L'Hopital's posed the question to Leibniz, what would the result be if n = 1/2. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." In these words, fractional calculus was born. Fractional Calculus [8]. used to solve many problems that applied to science and engineering (physical fields) with many methods, one of them fractional diffusion which have been applied in this paper to solve heat equation which can be divine as follows. The diffusion equation is a partial differential equation which depicts density dynamics in a material undergoing diffusion. The modified diffusion equations are used to portray some processes exhibiting diffusive-like behavior, which have a broad range of applications in mathematical physics, integral system and fluid mechanics. Thence, lots of methods have been used to solve this type of equations. In recent years, the fractional derivative with derivative of arbitrary orders have been developed to handle with problems in many areas, such as physics, applied mathematics, engineering and so forth. The fractional derivative can be defined in many forms, such as the Caputo derivative, the Riemann-Liouville derivative [5], the Gr¨unwald-Letnikov derivative and so on . However, most of them do not deal with the non-differentiable functions defined on Cantor sets. The local fractional derivative describes[5] the non-differential problems defined on Cantor sets, while the classical derivative and most fractional derivative deal with functions in Euclidean space, the theory has been successfully applied in investigating equations in fractal-like media, for example, the Navier-Stokes equations, the Helmholtz equations and the diffusion equations [5] Many methods have also developed to deal with the local fractional differential equations, such as the local fractional function decomposition method, the differential transform approach, the local fractional the variational iteration transform method and so on. In this study, our aim is to apply fractional Laplace trance form method and Fourier trance form method to solve the heat diffusion equations with fractional derivative and integral .

Methods: In this paper we used Laplace transform method and Fourier transform method with respect to Mittag Leffler integral equation.

2. The solution of stander heat equation:

The standard heat equation is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ .For this equation the temperature represented by *u*, which is a function of time *t*, and space *x*. [4],in this case we used the derivative operator *D* to write the heat equation with the following notation

$$D_t u = D_x^2 u. \qquad (1).$$

Consider an ideal one-dimensional rod of length *L* with the boundary conditions

$$u(t,0) = u(t,L) = 0 \quad , t \geq 0 . \qquad (2).$$

$$u(x,o) = f(x) . \qquad (3).$$

the solution is assumed to be of the form u(x,t) = X(x)T(t) plugging this in to equation (1) we obtain :

$$u_t(t,x) = T'(t)X(x) . \qquad (4) .$$

$$u_{xx}(t,x) = T\ (t)X^{''}(x) . \qquad (5) .$$

yields

$$T't)X(x) = T\ (t)X^{''}(x) . \qquad (6) .$$

Since this equation is separable there exists a constant of separation $C \in R$, such that

$$\frac{T'(t)}{T\ (t)} = \frac{X^{''}(x)}{X(x)} = \mathrm{C} . \qquad (7).$$

To obtain a non-trival real solution, assume C > 0 , this will satisfy the property of temperature decay to the defined boundary conditions. Thus we let C = -λ^2 for $\lambda \in$ R and we must solve the ordinary differential equations (ODEs) :

$$X^{''}(x) = -\lambda^2 \mathrm{X(x)} . \qquad (8).$$

$$X^{''}(x) + \lambda^2 \mathrm{X(x)} = 0 . \qquad (9).$$

equation (9) is homogeneous differential equations of second order. Suppose $\mathrm{X(x)} = \mathrm{e}^{mx}$ then $X''(x) = \mathrm{m}^2\ \mathrm{e}^{mx}$

by substitution the above values in equation (9) we obtain:

$$\mathrm{m}^2 + \lambda^2 = 0$$
$$\mathrm{m}^2 = -\lambda^2$$

$m = \mp \lambda \mathrm{i}$

for this we obtain $\mathrm{X(x)} = c_1\ \mathrm{e}^{\lambda \mathrm{ix}} + c_2\ \mathrm{e}^{-\lambda \mathrm{ix}}$

this (ODEs) have the solutions

$$\mathrm{X(x)} = c_1 cos\lambda\mathrm{x} + c_2\ \sin\lambda\mathrm{x}\ . \qquad (10)\ .$$

And $T'(t) + \lambda^2 T(t) = 0$. (11) .

suppose $\mathrm{T(t)} = \mathrm{e}^{mx}$ then $T'(t) = m\ \mathrm{e}^{mx}$ by substitution the values in equa- tion (11) we obtain :

$$m\ \mathrm{e}^{mx} + \lambda^2\ \mathrm{e}^{mx} = o$$
$$m + \lambda^2 = 0$$
$$m = -\lambda^2$$

Then $T(t) = c_3\ \mathrm{e}^{-\lambda^2 t}$

Preassigning the constant leads to general solution to heat equation (1)

$$\mathrm{u(t,x)} = A\cos(\lambda\mathrm{x})\ \mathrm{e}^{-\lambda^2 t} + B\sin(\lambda\mathrm{x})\ \mathrm{e}^{-\lambda^2 t}\ . \qquad (12).$$

where A= $C_1 C_3$, B= $C_2 C_3$

We satisfying the boundary condition:

secondLY moved on to the particular solution and started by satis- fying the boundary conditions (2).By inspection, the condition $u(t,0) = 0$ requires $A = 0$. The only non-trivial way to satisfy $u(t,L) = 0$ is by solving $\sin(\lambda L) = 0$.

Thus

$\lambda_n = \frac{n\pi}{L}$ *we find that :*

$$\mathrm{u(t,x)} = \mathrm{X(x)\ T(t)} = B\ \sin(\lambda_n\ \mathrm{x})\ \mathrm{e}^{-(\frac{n\pi}{L})^2 t}. \qquad (13).$$

$$u_n(t,x) = X(x)T(t) = B_n\ \sin\left(\frac{n\pi}{L}\ x\right)\ e^{-(\frac{n\pi}{L})^2 t}\ . \qquad (14).$$

for $n \in \mathrm{N}$, yields infinitely many solutions to the heat equation (The sum of such solutions is given by the Fourier series solution) by using the properties of sine and cosine and reached to this particular solution.

$$u_n(x,t) = \sum_{n=1}^{\infty} B_n\ \sin\left(\frac{n\pi}{L}\ x\right) e^{-(\frac{n\pi}{L})^2 t}\ . \qquad (15).$$

By satisfying

$$\mathrm{u(0,x)} = \mathrm{f(x)} = \sum_{n=1}^{\infty} B_n\ \sin\left(\frac{n\pi}{L}\ x\right). \qquad (16).$$

The B_n can be calculated from

$$B_n = \frac{2}{l}\int_0^t f(x)\sin nx\,dx$$

5.3 The solution of fractional heat diffusion

To solve fractional heat diffusion

$$D_t^{\propto} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}. \qquad (17).$$

With initial value :

$$u(x,0)= f(x) \qquad ,(|x|<\infty, t>0). \qquad (18).$$

$$\lim_{x\mp\infty} u(x,t) = 0 \quad ,(t>0). \qquad . \qquad (19).$$

When $0 <\propto< 1$ we take Laplace transform to x we obtain

$$\int \{D_t^{\propto} f(t), s\} = s^{\propto} F(s) - s^{\propto-1} f(0). \qquad (20).$$

Applied (17),(18),(19) in (20) yields

$$s^{\propto}\tilde{u}(x,t) - s^{\propto-1} f(x) = \frac{\partial^2 \tilde{u}(x,s)}{\partial x^2} \qquad ,(|x|<\infty). \qquad (21).$$

$$\lim_{x\mp\infty} \tilde{u}(x,s) = 0 \qquad ,t > 0. \qquad (22).$$

applying Fourier exponential transform to t in equation(21) and

utilizing the boundary conditions (22) we obtain

$$\tilde{\tilde{u}}(\beta,s)=\frac{s^{\propto-1}}{s^{\propto}+\beta^2} F(\beta). \qquad (23).$$

where $\tilde{\tilde{u}}$ (β,s) and F (β) are the Fourier transform of $\tilde{u}(x,s)$

is $\frac{s^{\propto-1}}{s^{\propto}+\beta^2}$ fraction and f(x).the inverse Laplace transform of mittag-leffler function in tow pa- .

(where$E_{\propto,1}$ $E_{\propto,1}(\beta^2 t^{\propto})$

rameters) there for in varsion Fourier and the Laplace transform gives the solution in form:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi. \qquad (24).$$

$$G(x,t) = \frac{1}{\pi}\int_0^{\infty} E_{\propto,1}(\beta^2 t^{\propto}) \cos(\beta x^{\propto}) d\beta.$$

$$= t^{-p}\, w(-z, -p, 1-p). \qquad (25).$$

. Where w (z, p,1-p) is the wright function

In order to get special solution, we used many values of ∝

In stander fractional heat diffusion

$$D_t^{\propto} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

Case 1. at ∝ = 1/2 it takes the form

$$D_t^{1/2} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}. \qquad (26).$$

The boundary conditions :

$$u(x,0)= f(x) \qquad ,(|x|<\infty, t>0). \qquad (27).$$

$$\lim_{x\mp\infty} u(x,t) = 0 \quad ,(t>0). \qquad . \qquad (28).$$

When we take Laplace transform to t we obtain :

$$\mathcal{L}\{D_t^{\frac{1}{2}} f(t), s\} = s^{1/2}\, F(s) - s^{\frac{1}{2}-1}\, f(0).$$
$$= s^{1/2}\, F(s) - s^{-1/2}\, f(0). \qquad (29).$$

Applied (26),(27),(28) in (29) yields

$$s^{\frac{1}{2}}\tilde{u}(x,t) - s^{-1/2}\, f(x) = \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} \quad ,(|x|<\infty). \qquad (30).$$
$$\lim_{x\mp\infty} \tilde{u}(x,s) = 0 \quad ,t > 0. \qquad (31).$$

applying Fourier exponential transform to x in equation(30) and utilizing the boundary conditions (31) we obtain

$$\tilde{\tilde{u}}\,(\beta,s) = \frac{s^{-1/2}}{s^{1/2}+\beta^2}\, F(\beta). \qquad (5.3.14).(32).$$

where $\tilde{\tilde{u}}$ (β,s) and F (β) are the Fourier transform of $\tilde{u}(x,s)$

is $\frac{s^{-1/2}}{s^{1/2}+\beta^2}$ fraction and f(x).the inverse Laplace transform of is mittag-leffler function in tow parmeters)

there for in varsion . (where$E_{\frac{1}{2},1}$ $E_{\frac{1}{2},1}(\beta^2 t^{\frac{1}{2}})$

Fourier and the Laplace transform gives the solution in form:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi. \qquad (33).$$
$$G(x,t) = \frac{1}{\pi}\int_0^{\infty} E_{\frac{1}{2},1}\left(\beta^2 t^{\frac{1}{2}}\right)\cos(\beta x) d\beta. \qquad (34).$$

$t^{-p}\, w(-z\,,-p\,,1-p) =$

. Where w (z, p,1-p) is the wright function

Case 2. $D_x^{\propto} u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}$

at $\propto = 1/2$ it takes the form

$$D_x^{\frac{1}{2}} u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}. \qquad (35).$$

The boundary conditions :

$$u(0,t) = f(x) \quad ,(|x|<\infty, t>0). \qquad (36).$$
$$\lim_{x\mp\infty} u(x,t) = 0 \quad ,(t>0). \qquad (37).$$

When we take Laplace transform to x we obtain :

$$\mathcal{L}\{D_x^{\frac{1}{2}} f(x,s)\} = s^{1/2}\, F(s) - s^{\frac{1}{2}-1}\, f(0).$$
$$= s^{1/2}\, F(s) - s^{-1/2}\, f(0). \qquad (38).$$

Applied (35),(36),(37) in (38) yields

$$s^{\frac{1}{2}}\tilde{u}(x,s) - s^{-1/2}\, f(t) = \frac{\partial^2\, \tilde{u}(x,t)}{\partial t^2} \quad ,(|x|<\infty). \qquad (39).$$

$$\lim_{x\mp\infty} \tilde{u}(x,s) = 0 \quad ,t > 0. \qquad (40).$$ applying Fourier exponential transform to t in equation(5.3.20) and utilizing the boundary conditions (40) we obtain

$$\tilde{\tilde{u}}(\beta,s) = \frac{s^{-1/2}}{s^{1/2}+\beta^2}\ F(\beta). \qquad (41).$$

where $\tilde{\tilde{u}}(\beta,s)$ and F (β) are the Fourier transform of $\tilde{u}(t,s)$ and f(x). The inverse Laplace transform of fraction $\frac{s^{-1/2}}{s^{1/2}+\beta^2}$ is $E_{1,\frac{1}{2}}(\propto^2\ t)$. (where $E_{1,\frac{1}{2}}$ is mittag-leffler function in tow parameters) there for in varsion Fourier and the Laplace transform gives the solution in form:

$$u\ (x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi. \qquad (42).$$

$$G\ (x,t) = \frac{1}{\pi}\int_0^{\infty} E_{\frac{1}{2},1}(\beta^2 t)\cos(\beta x) d\beta. \qquad (43).$$

$t^{-p}\ w(-z\ ,-p\ ,1-p) =$

. Where w (z, p,1-p) is the wright function

Case 3.

$$D_t^{\propto}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

at $\propto = 3/2$ it takes the form

$$D_t^{3/2}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}\ . \qquad (44).$$

The boundary conditions :

$$u(x,0)= f(x) \qquad , (|x|<\infty, t>0)\ . \qquad (45).$$

$$\lim_{x\mp\infty} u(x,t) = 0 \quad ,(t>0). \qquad (46).$$

When we take Laplace transform to t we obtain:

$$\mathcal{L}\{D_t^{\frac{3}{2}}f(t,.s)\} = s^{3/2}\ F(s) - s^{\frac{3}{2}-1}\ f(0).$$

$$= s^{3/2}\ F(s) - s^{1/2}\ f(0). \qquad (47).$$

Applied(44),(45a),(46) in (47). yields

$$s^{\frac{3}{2}}\tilde{u}(x,t) - s^{1/2}\ f(x) = \frac{\partial^2\tilde{u}(x,t)}{\partial x^2} \qquad , (|x|<\infty). \qquad (48).$$

$$\lim_{x\mp\infty} \tilde{u}(x,s) = 0 \quad ,t > 0. \qquad (49).$$

applying Fourier exponential transform to x in equation(48) and utilizing the boundary conditions (49)we obtain

$$\tilde{\tilde{u}}\ (\beta,s) = \frac{s^{1/2}}{s^{3/2}+\beta^2}\ F(\beta)\ . \qquad (50).$$

where $\tilde{\tilde{u}}(\beta,s)$ and F (β) are the Fourier transform of $\tilde{u}(x,s)$ and f(x). The inverse Laplace transform of fraction $\frac{s^{1/2}}{s^{3/2}+\beta^2}$ is $E_{\frac{3}{2},1}(\beta^2 t^{\frac{3}{2}})$. (where

$E_{\frac{3}{2},1}$ is Mittag-leffler function in tow parameters) there for in varsion Fourier and the Laplace transform gives the solution in form:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi. \quad (51).$$

$$G(x,t) = \frac{1}{\pi}\int_{0}^{\infty} E_{\frac{3}{2},1}\left(\beta^2 t^{\frac{3}{2}}\right)\cos(\beta x) d\beta. \quad (52).$$

$t^{-p}\, w(-z, -p, 1-p) =$

. Where w (z, p,1-p) is the wright function

Case. 4 $D_x^{\propto} u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}$ When $\propto = 3/2$

$$D_x^{\frac{3}{2}} u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} \quad . \quad (53).$$

The boundary conditions :

$$u(0,t) = f(t) \quad , (|x|<\infty, t>0). \quad (54).$$

$$\lim_{x \mp \infty} u(x,t) = 0 \quad , (t>0). \quad (55).$$

When we take Laplace transform to x we obtain

$$\mathcal{L}\{D_x^{\frac{3}{2}} f(x,.s)\} = s^{3/2} F(s) - s^{\frac{3}{2}-1} f(0).$$

$$= s^{3/2} F(s) - s^{1/2} f(0). \quad (56).$$

Applied (53),(54),(55) in(56) yields

$$s^{\frac{3}{2}} \tilde{u}(s,t) - s^{1/2} f(t) = \frac{\partial^2 \tilde{u}(x,t)}{\partial t^2} \quad , (|x|<\infty). \quad (57).$$

$$\lim_{x \mp \infty} \tilde{u}(s,t) = 0 \quad , t > 0. \quad (58).$$

applying Fourier exponential transform to t in equation (57) and utilizing the boundary conditions (58) we obtain:

$$\tilde{\tilde{u}}(\beta,s) = \frac{s^{1/2}}{s^{3/2}+\beta^2} F(\beta). \quad (59).$$

where $\tilde{\tilde{u}}(\beta,s)$ and F (β) are the Fourier transform of $\tilde{u}(x,s)$ and f(x). the inverse Laplace transform of the fraction $\frac{s^{1/2}}{s^{3/2}+\beta^2}$ is

$E_{1,\frac{3}{2}}(\beta^2 t^{\frac{3}{2}})$. (where $E_{1,\frac{3}{2}}$ mittag-leffler function in tow pa rameters)

there for in varsion Fourier and the Laplace transform gives the solution in form:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi. \quad (60).$$

$$G(x,t) = \frac{1}{\pi}\int_{0}^{\infty} E_{\frac{3}{2},1}\left(\beta^2 t^{\frac{3}{2}}\right)\cos(\beta x) d\beta. \quad (61).$$

$$= t^{-p}\, w(-z, -p, 1-p).$$

. Where w (z, p,1-p) is the wright function

3. Discussion:

Now the theory of fractional differential equation is very important in many applications in this study we used the Laplace and Fourier method to solve heat diffusion. We anticipate that is possible to solve many partial fractional differential equations like wave equation, Schrodinger equation and Burger equation.

4. Conclusion:

We conclude that the fractional heat equation is a physically legitimate generalization of the standard heat equation that might be used for values $\alpha \approx 1$. As expected all solutions sufficiently close to α satisfy the boundary conditions and display physically realistic properties.We anticipate many partial differential equations in classical form can be solved in fractional form for example wave equation , Schrodinger equation and Burger equation

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الملخص :

في هذه الورقة تمت دراسة معادلة انتشار الحرارة باستخدام التفاضل الكسري وكان التركيز على البارومترات ∝و $\beta = 1/2$ وحصلنا على الحل التحليلي باستخدام طرق لابلاس و فوريير وقد كان هذا الحل متقاربا للحل الكلاسيكي (حل انتشار الموجة الحرارية).

الكلمات المفتاحية : التفاضل والتكامل الجزئي - معادله الحرارة - فوريه وتحويل لابلاس
