Journal of Nature, Life and Applied Sciences (JNSLAS) • Vol 9, Issue 1 (2025) • P: 51 – 38

https://journals.ajsrp.com/index.php/jnslas

ISSN: 2522-3356 (Online) • ISSN: 2522-3356 (Print)

Maxwell's Equations Under Lorentz Transformations: Investigating the Spontaneous Emergence of Magnetic Charge and Current Densities

Mr. Louai Hassan Elzein Bashier

Sudan

Received: 30/12/2024

Revised: 19/01/2025

Accepted: 08/02/2025

Published: 15/03/2025

* Corresponding author: Louai.Elzein@proton.me

Citation: Bashier, L. H. (2024). Maxwell's Equations Under Lorentz Transformations: Investigating the Spontaneous Emergence of Magnetic Charge and Current Densities. *Journal of natural sciences, life and applied sciences, 9*(1), 38 – 51. https://doi.org/10.26389/AJSRP.K010125

Abstract: The quest for magnetic monopoles has captivated theoretical physicists due to their potential to reshape our understanding of electromagnetism. Despite extensive research, definitive mathematical proof of their existence remains elusive. This paper presents a rigorous mathematical framework supporting the existence of magnetic monopoles within the covariant formulation of classical electromagnetism and special relativity. Utilizing tensor calculus and the covariance form of Maxwell's equations, we derive expressions for magnetic charge density and magnetic current density through Lorentz transformations. Our approach emphasizes the expansion of tensorial equations, particularly focusing on the covariant derivatives of the electromagnetic field tensor and its dual. The results demonstrate the recovery of magnetic charge and current densities in Maxwell's equations, providing a theoretical foundation for the existence of magnetic monopoles. A magnetic monopole model is introduced, establishing the relationship between the electric field, the monopole's magnetic charge, and their dependence on the observer's relative velocity as a direct consequence of the modified Gauss's law for magnetism. This model suggests the existence of dipole bosons—electric or magnetic—which function analogously to the Higgs boson. These dipole bosons are proposed to confer mass, charge, and spin to charged particles, whether they are electric or magnetic in nature.

Keywords: Special relativity, Lorentz transformation, Electromagnetic interactions, Particle-theory models, Magnetic monopoles, Higgs boson.

معادلات ماكسويل تحت تحولات لورنتز: دراسة الظهور التلقائي لكثافات الشحنة والمجال المغناطيسي

أ. لؤي حسن الزين بشير

السودان

المستخلص: لقد أسرت مسألة وجود أحادية المغناطيس الفيزيائيين النظريين بسبب إمكانياتها في إعادة تشكيل فهمنا للكهرومغناطيسية. على الرغم من الأبحاث المكثفة، لا يزال إثبات رياضي قاطع لوجودها بعيد المنال. يقدم هذا البحث إطارًا رياضيًا صارمًا يدعم وجود أحادية المغناطيس ضمن الصيغة التغايرية للكهرومغناطيسية الكلاسيكية والنظرية النسبية الخاصة. من خلال استخدام حساب الموترات والصيغة التغايرية لمعادلات ماكسويل، أُستنتِجت تعبيرات لكثافة الشحنة المغناطيسية وكثافة التيار المغناطيسي من خلال تحولات لورنتز. يركز نهجنا على فك مجموعة من المعادلات الموترية، مع التركيز بشكل خاص على المشتقات التغايرية لموتر المجال الكهرومغناطيسي وموتره الثنائي. تُظهر النتائج استعادة كثافات الشحنة والتيار المغناطيسي في معادلات ماكسويل، مما يوفر أساسًا نظريًا لوجود أحادية المغناطيس. تم تقديم نموذج أحادي القطب المغناطيسي والذي يُحدد العلاقة بين المجال الكهربائي، الشحنة المغناطيسية لأحادي القطب، وأعتمادهما على السرعة النسبية للمراقب كنتيجة مباشرة لقانون غاوس المعدل للمغناطيسية. يقترح هذا النموذج وجود بوزونات ثنائية القطب—كهربائية أو مغناطيسية—تعمل بشكل مشابه لبوزون هيغز. هذه البوزونات الثنائية القطب المقترحة تمنح الجسيمات المشحونة—سواء كانت كهربائية أو مغناطيسية—الكتلة، الشحنة، والغزل.

الكلمات المفتاحية: النسبية الخاصة، تحويلات لورنتز، التفاعلات الكهرومغناطيسية، نماذج نظرية الجسيمات، أُحادي القطب المغناطيسي، بوزون هيغز.

1. Introduction

In 1931, physicist Paul Dirac theorized the existence of magnetic monopoles – particles possessing only a single magnetic pole (north or south). This groundbreaking concept, while yet to be experimentally proven, has profoundly influenced several areas of physics. Dirac demonstrated that the existence of magnetic monopoles could elegantly explain the observed quantization of electric charge (Dirac, 1931). This pivotal insight propelled the exploration of magnetic monopoles into the realms of quantum field theory, grand unified theories (GUTs) that aim to unify the fundamental forces of nature, and the study of the universe's origins (cosmology). Dirac monopoles challenged classical electromagnetism by carrying a net magnetic charge. Dirac showed that monopoles could exist without violating Maxwell's equations if incorporated into a modified framework. He derived the quantization condition $eg = n\hbar c/2$, where e is the elementary electric charge (the smallest unit of electric charge, such as that of an electron), g is the magnetic charge of the monopole, n is an integer representing the quantization condition, $\hbar$ is the reduced Planck constant, and c is the speed of light in a vacuum (Dirac, 1931). In his later work, Dirac expanded on this idea by developing a more comprehensive theory that included the interaction of magnetic poles with charged particles through the electromagnetic field. He introduced the concept of "strings" attached to magnetic poles, which are lines of singularity in the electromagnetic potentials, and showed that the quantization condition $eg = n\hbar c/2$ must hold for the theory to be consistent (Dirac, 1948).

In the 1970s, 't Hooft and Polyakov expanded on Dirac's work by demonstrating that magnetic monopoles naturally arise in grand unified theories (GUTs). Their independent discoveries showed that monopoles emerge as topological solitons in non-Abelian gauge theories, such as the Georgi-Glashow model, when the symmetry is spontaneously broken ('t Hooft, 1974; Polyakov, 1974). These 't Hooft-Polyakov monopoles possess finite mass determined by the symmetry-breaking scale, marking them as stable and theoretically significant objects. GUT monopoles are typically characterized by extremely high masses in the range of 10^{14}– 10^{16} GeV, which makes their detection infeasible in particle colliders. However, recent advancements in extensions of the Standard Model, such as the Born-Infeld modifications and topological adjustments in the Higgs sector, predict monopoles with much lower masses, in the range of a few TeV, thus making them accessible for detection in current and future colliders.

The study of monopoles advanced further with developments in quantum field theory and supersymmetric (SUSY) frameworks. In supersymmetric theories, monopoles often manifest as BPS (Bogomol'nyi-Prasad-Sommerfield) states, characterized by their stability and minimal energy configuration (Prasad & Sommerfield, 1975). The Seiberg-Witten solution of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory offered groundbreaking insights, showing that monopoles could become massless at specific points in the moduli space. This dual description recast the theory in terms of magnetic variables and highlighted monopole dynamics' role in non-perturbative phenomena (Seiberg & Witten, 1994). In $\mathcal{N} = 2$ supersymmetric QCD (SQCD), monopoles are central to the confinement mechanism, where their condensation confines chromoelectric flux, binding quarks into hadrons. This behavior is analogous to the Meissner effect in superconductors, where magnetic flux is confined to vortices (Seiberg & Witten, 1994). These findings underline the crucial role of monopoles in understanding confinement and dynamical symmetry breaking (Konishi, 2007).

Magnetic monopoles also play a significant role in cosmological models. According to GUTs, monopoles would have been abundantly produced in the early universe. However, this overproduction, known as the "monopole problem," posed a major challenge for cosmology (Preskill, 1979). Inflationary cosmology, introduced by Guth (1981), resolved this issue by proposing a rapid expansion of the universe, diluting monopole density to undetectable levels.

In condensed-matter systems, "magnetic monopoles" refer to emergent quasiparticles that mimic monopole-like behavior, observed in materials such as spin ices (e.g., dysprosium titanate). These quasiparticles arise from collective magnetic excitations, forming defects analogous to Dirac strings, with ends acting as monopole-like charges (Castelnovo et al., 2008; Morris et al., 2009). Experiments have demonstrated "magnetricity," where magnetic charge currents may measure (Giblin et al., 2011), and synthetic monopoles have been created in Bose-Einstein condensates (Ray et al., 2014). While not true elementary monopoles, these systems provide insights into monopole-like phenomena in quantum materials.

Experimentally, extensive searches have been conducted for magnetic monopoles in cosmic rays, geological materials, and particle accelerators. Early experiments, such as Cabrera's use of superconducting detectors, set upper bounds on monopole flux (Cabrera, 1982). More recent efforts, including the MACRO Collaboration's cosmic ray studies, further constrained monopole abundance (MACRO Collaboration, 2002). High-energy experiments, such as those at CERN's Large Hadron Collider (LHC), aim to detect monopoles directly through particle collisions, but no conclusive evidence has yet been found. Also, advances in quantum sensors, such as

Superconducting Quantum Interference Devices (SQUIDs), have greatly improved sensitivity to weak magnetic fields, enhancing magnetic monopole searches (Fagaly, 2006). The MoEDAL experiment at CERN employs SQUID magnetometers to analyze aluminum trapping volumes for monopoles (Mavromatos & Mitsou, 2020). Scanning SQUID microscopy has also been proposed to detect monopoles in spin ice materials due to its high sensitivity (Kirschner et al., 2018). These innovations refine detection limits and improve the sensitivity to detect magnetic monopoles.

Dirac's hypothesis remains central to theoretical physics, explaining electric charge quantization and inspiring research on unification and the universe's structure. Monopole discovery would revolutionize our understanding of fundamental forces, with implications for quantum gravity, black hole physics, and early universe studies. And this paper seeks to present a rigorous mathematical derivation of the Lorentz-transformed Maxwell's equations and to establish, with mathematical certainty, the theoretical existence of magnetic charge density (i.e., magnetic monopoles) and magnetic current density, and to introduce a modified theoretical framework integrating electromagnetic theory with relativistic mechanics, which can then be approximately reduced to a combination of electromagnetic theory and classical mechanics. It incorporated the uniform motion at its two levels, relativistic or nonrelativistic, into the body of electromagnetism.

2. General forms of covariant derivative of the transformed electromagnetic field tensors and stress–energy tensor

The Lorentz transformation and Maxwell's equations are fundamental components of classical and modern physics, intricately linked through the theory of special relativity. The Lorentz transformation, developed by Hendrik Lorentz in the late 19th century, provides the mathematical framework for how measurements of space and time by two observers are related when the observers are in uniform relative motion (Lorentz, 1904). These transformations were initially motivated by attempts to explain the invariance of Maxwell's equations in different inertial reference frames, particularly in the context of the Michelson-Morley experiment (Michelson & Morley, 1887), which failed to detect any motion relative to the supposed luminiferous ether.

Maxwell's equations, formulated by James Clerk Maxwell, describe the behavior of electric and magnetic fields and how they propagate and interact with charges and currents (Maxwell, 1865). The equations unify electricity, magnetism, and optics into a single theoretical framework, predicting the existence of electromagnetic waves traveling at the speed of light. This remarkable prediction highlighted the fundamental nature of light as an electromagnetic phenomenon and raised questions about the underlying principles governing space and time.

The synthesis of Maxwell's equations with the principles of relativity culminated in Albert Einstein's theory of special relativity (Einstein, 1905), which discarded the notion of the ether and established that the laws of physics, including the speed of light, are invariant in all inertial frames. Einstein demonstrated that the Lorentz transformations naturally arise from the principle of relativity and the constancy of the speed of light, fundamentally altering the classical understanding of space and time.

In the framework of special relativity, Maxwell's equations are elegantly expressed in their covariant form using tensor notation. The electromagnetic field is represented by the antisymmetric electromagnetic field tensor $F^{\alpha\beta}$, which encapsulates the electric and magnetic fields into a unified mathematical entity. The tensor transforms consistently under Lorentz transformations, reflecting the relativistic nature of electromagnetism. In this formalism, the four Maxwell's equations reduce to two compact tensor equations:

$$\frac{\partial}{\partial x^{\alpha}} F^{\alpha\beta} = \mu_0 J^{\beta} \quad \text{, and} \quad \frac{\partial}{\partial x^{\alpha}} G^{\alpha\beta} = 0. \qquad (1)$$

where J^{β} is the four-current vector, encompassing charge and current densities and $G^{\alpha\beta}$ is the dual contravariant electromagnetic field tensor. These equations not only demonstrate the unification of electric and magnetic fields but also exemplify the compatibility of electromagnetism with the principles of relativity. The invariance of $F^{\alpha\beta}$ under Lorentz transformations underscores the deep connection between the geometry of spacetime and the dynamics of the electromagnetic field, marking a cornerstone of modern theoretical physics.

Another significant quantity in the field of electromagnetism is the electromagnetic stress-energy tensor, $T^{\alpha\beta}$, which plays a central role in the covariant formulation of electromagnetism, as it encapsulates the energy density, momentum density, and stress (or pressure) associated with the electromagnetic field. This tensor is a rank-2 symmetric tensor that serves as the energy-momentum tensor for the electromagnetic. Its components describe the distribution of energy, momentum, and stresses of the electromagnetic field in spacetime.

The transformation laws for the electromagnetic field tensor $F^{\alpha'\beta'}$, its dual tensor $G^{\alpha'\beta'}$, and the electromagnetic stress-energy tensor $T^{\alpha'\beta'}$ are fundamental to ensuring the covariance of physical laws in theory of relativity. These tensors represent the electric and magnetic fields, their dual relationships, and the energy-momentum properties of the electromagnetic field, respectively. Under a Lorentz transformation Λ^{μ}_{ν} , the components of these tensors transform according to the following laws:

$$F^{\alpha'\beta'} = \Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}F^{\alpha\beta},\ G^{\alpha'\beta'} = \Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}G^{\alpha\beta}\ \text{, and}\quad T^{\alpha'\beta'} = \Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}T^{\alpha\beta}. \tag{2}$$

Here, $\Lambda^{\alpha'}_{\alpha}$ represents the Lorentz transformation matrix, which depends on the relative velocity and orientation of the reference frames. The electromagnetic field tensor $F^{\alpha\beta}$ encodes the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, and its transformation reflects how these fields mix under changes in an observer's motion. Similarly, the dual tensor $G^{\alpha\beta}$, defined as $G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}$, represents the complementary magnetic and electric field components and transforms analogously. The stress-energy tensor $T^{\alpha\beta}$ encapsulates the energy density, momentum density, and stresses of the electromagnetic field. Its transformation law ensures that quantities such as energy flux and momentum density remain consistent across inertial frames. These transformation laws arise naturally from the principles of Lorentz invariance, ensuring that Maxwell's equations and the energy-momentum relations retain their form across all inertial observers, a cornerstone of modern physics. Applying Equations (1) to the transformed tensors (Equations (2)) accompany with replacing partial derivative with covariant derivative, yields:

$$F^{\alpha'\beta'}{}_{;\alpha'} = \frac{\partial}{\partial x^{\alpha'}}F^{\alpha'\beta'} + \Gamma^{\alpha'}_{\gamma'\alpha'}F^{\gamma'\beta'} + \Gamma^{\beta'}_{\gamma'\alpha'}F^{\alpha'\gamma'}, \tag{3}$$

and the general form of covariant derivative of the transformed electromagnetic field tensor becomes:

$$F^{\alpha'\beta'}{}_{;\alpha'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}F^{\alpha\beta} + \frac{\partial x^{\rho}}{\partial x^{\gamma'}}\Gamma^{\alpha}_{\rho\alpha}\Lambda^{\gamma'}_{\gamma}\Lambda^{\beta'}_{\beta}F^{\gamma\beta} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\alpha'}}\Lambda^{\gamma'}_{\gamma}\Lambda^{\beta'}_{\beta}F^{\gamma\beta} + \frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x^{\gamma'}}\frac{\partial x^{\sigma}}{\partial x^{\alpha'}}\Gamma^{\beta}_{\rho\sigma}\Lambda^{\alpha'}_{\alpha}\Lambda^{\gamma'}_{\gamma}F^{\alpha\gamma} + \frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial^2 x^{\beta}}{\partial x^{\alpha'}\partial x^{\gamma'}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\gamma'}_{\gamma}F^{\alpha\gamma} \tag{4}$$

Similarly, the general forms of covariant derivative of the transformed dual electromagnetic field tensor and the electromagnetic stress-energy tensor:

$$G^{\alpha'\beta'}{}_{;\alpha'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}G^{\alpha\beta} + \frac{\partial x^{\rho}}{\partial x^{\gamma'}}\Gamma^{\alpha}_{\rho\alpha}\Lambda^{\gamma'}_{\gamma}\Lambda^{\beta'}_{\beta}G^{\gamma\beta} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\alpha'}}\Lambda^{\gamma'}_{\gamma}\Lambda^{\beta'}_{\beta}G^{\gamma\beta} + \frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x^{\gamma'}}\frac{\partial x^{\sigma}}{\partial x^{\alpha'}}\Gamma^{\beta}_{\rho\sigma}\Lambda^{\alpha'}_{\alpha}\Lambda^{\gamma'}_{\gamma}G^{\alpha\gamma} + \frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial^2 x^{\beta}}{\partial x^{\alpha'}\partial x^{\gamma'}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\gamma'}_{\gamma}G^{\alpha\gamma} \tag{5}$$

$$T^{\alpha'\beta'}{}_{;\alpha'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}T^{\alpha\beta} + \frac{\partial x^{\rho}}{\partial x^{\gamma'}}\Gamma^{\alpha}_{\rho\alpha}\Lambda^{\gamma'}_{\gamma}\Lambda^{\beta'}_{\beta}T^{\gamma\beta} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'}\partial x^{\alpha'}}\Lambda^{\gamma'}_{\gamma}\Lambda^{\beta'}_{\beta}T^{\gamma\beta} + \frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x^{\gamma'}}\frac{\partial x^{\sigma}}{\partial x^{\alpha'}}\Gamma^{\beta}_{\rho\sigma}\Lambda^{\alpha'}_{\alpha}\Lambda^{\gamma'}_{\gamma}T^{\alpha\gamma} + \frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial^2 x^{\beta}}{\partial x^{\alpha'}\partial x^{\gamma'}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\gamma'}_{\gamma}T^{\alpha\gamma} \tag{6}$$

Equations (4) and (5) are the general transformation laws of Maxwell's equations and Equation (6) is the general transformation law of covariant derivative of electromagnetic stress–energy tensor.

3. Lorentz transformation of Maxwell's equations in flat spacetime

Maxwell's equations form the foundation of classical electromagnetism, describing the interplay between electric and magnetic fields and their sources. In flat spacetime, the framework of special relativity, these equations are expressed in a way that respects the principles of relativity, ensuring their validity for all observers moving at constant velocities. They explain how electric charges generate electric fields, how currents and changing electric fields produce magnetic fields, and how these fields propagate as electromagnetic waves, such as light, at the speed of light in a vacuum. In this relativistic context, electric and magnetic fields are understood as components of a single electromagnetic field, unified in a way that highlights the deep connection between space and time. This formulation is essential for modern physics, influencing everything from quantum electrodynamics to the study of electromagnetic waves in various media. Classical Maxwell's equations are special case of Equations (4) and (5) where all the affinities ($\Gamma^{\alpha\beta}_{\gamma}$) vanishes due to flatness of spacetime, that is,

$$\frac{\partial}{\partial x^{\alpha'}}F^{\alpha'\beta'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}F^{\alpha\beta}, \tag{7}$$

$$\frac{\partial}{\partial x^{\alpha'}}\boldsymbol{G}^{\alpha'\beta'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\boldsymbol{\Lambda}^{\alpha'}{}_{\alpha}\boldsymbol{\Lambda}^{\beta'}{}_{\beta}\boldsymbol{G}^{\alpha\beta}, \tag{8}$$

3.1 Lorentz transformed Gauss's laws of electricity and of magnetism

Gauss's laws are two of the four fundamental equations that constitute Maxwell's equations, which describe the behavior of electric and magnetic fields. Gauss's law for electricity states that the electric flux through a closed surface is proportional to the total charge enclosed within that surface. This law highlights the relationship between electric charges and the electric fields they produce, emphasizing that electric field lines originate from positive charges and terminate at negative charges. On the other hand, Gauss's law for magnetism states that the net magnetic flux through any closed surface is zero, reflecting the fact that magnetic monopoles do not exist in nature. Instead, magnetic field lines form continuous loops, with no starting or ending points. Together, these laws provide a foundational understanding of how electric and magnetic fields behave in the presence of charges and currents, and they are essential for analyzing electrostatic and magnetostatic systems. Gauss's law for electricity can be obtained in a usual way from Equation (7) by substituting $\beta' = 0$, assuming the relative uniform motion u between the field inertial frame of reference, S-frame, and the observer inertial frame of reference, S'-frame, is in the x-direction, that is,

$$\frac{\partial}{\partial x^{\alpha'}}F^{\alpha'0} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}{}_{\alpha}\Lambda^{0}{}_{\beta}F^{\alpha\beta} \tag{9}$$

where:

$$(\Lambda^{\alpha'}{}_{\alpha})_u = \left(\Lambda^{\beta'}{}_{\beta}\right)_u = \begin{pmatrix} \gamma_u & -\gamma_u\frac{u}{c} & 0 & 0 \\ -\gamma_u\frac{u}{c} & \gamma_u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation of partial derivative can be obtained through chain rule; $\frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}$, and inverse Lorentz transformation; $ct = \gamma_u\left(ct' + \frac{u}{c}x'\right)$, $x = \gamma_u\left(x' + \frac{u}{c}ct'\right)$, $y = y'$, $z = z'$, where u is the relative velocity between frames in the x-direction, c is the speed of light, and $\gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$ is the Lorentz factor:

$$\begin{aligned}
&\left(\frac{\partial x^0}{\partial x'^0}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^0}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^0}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^0}\frac{\partial}{\partial x^3}\right) = \gamma_u\left(\frac{\partial}{\partial ct} + \frac{u}{c}\frac{\partial}{\partial x}\right),\\
&\left(\frac{\partial x^0}{\partial x'^1}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^1}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^1}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^1}\frac{\partial}{\partial x^3}\right) = \gamma_u\left(\frac{u}{c}\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right),\\
&\left(\frac{\partial x^0}{\partial x'^2}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^2}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^2}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^2}\frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial y}\ ,\\
&\left(\frac{\partial x^0}{\partial x'^3}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^3}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^3}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^3}\frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial z}\ .
\end{aligned} \tag{10}$$

Expanding the tensor Equation (7) using the electromagnetic field tensors:

$$(F^{\alpha'\beta'}) = \begin{pmatrix} 0 & -E'_{x'}/c & -E'_{y'}/c & -E'_{z'}/c \\ E'_{x'}/c & 0 & -B'_{z'} & B'_{y'} \\ E'_{y'}/c & B'_{z'} & 0 & -B'_{x'} \\ E'_{z'}/c & -B'_{y'} & B'_{x'} & 0 \end{pmatrix}\ \text{, and}\quad (F^{\alpha\beta}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial x^{\alpha'}}F^{\alpha'0} = \frac{\partial}{\partial x'^0}F'^{00} + \frac{\partial}{\partial x'^1}F'^{10} + \frac{\partial}{\partial x'^2}F'^{20} + \frac{\partial}{\partial x'^3}F'^{30} = \frac{1}{c}\boldsymbol{\nabla}'\cdot\mathbf{E}',$$

$$= \frac{\partial x^{\delta}}{\partial x^{\alpha'}} \frac{\partial}{\partial x^{\delta}} \Lambda^{\alpha'}_{\ \alpha} \Lambda^{0}_{\ \beta} F^{\alpha\beta},$$

$$= \gamma_u \left(\frac{\partial}{\partial ct} + \frac{u}{c}\frac{\partial}{\partial x}\right) \Lambda^{0}_{\ \alpha} \Lambda^{0}_{\ \beta} F^{\alpha\beta} + \gamma_u \left(\frac{u}{c}\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right) \Lambda^{1}_{\ \alpha} \Lambda^{0}_{\ \beta} F^{\alpha\beta} + \frac{\partial}{\partial y}\Lambda^{2}_{\ \alpha} \Lambda^{0}_{\ \beta} F^{\alpha\beta} + \frac{\partial}{\partial z}\Lambda^{3}_{\ \alpha} \Lambda^{0}_{\ \beta} F^{\alpha\beta}, \tag{11}$$

which reduces to:

$$\nabla' \cdot \mathbf{E}' = \gamma_u \left[\nabla \cdot \mathbf{E} - u\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + u\mu_0\epsilon_0 \frac{\partial E_x}{\partial t}\right]. \tag{12}$$

By repeating the process for y-direction and z-direction, we observe the following general pattern:

$$\nabla' \cdot \mathbf{E}' = \gamma_v \left[\nabla \cdot \mathbf{E} - v\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + v\mu_0\epsilon_0 \frac{\partial E_y}{\partial t}\right],$$

$$\nabla' \cdot \mathbf{E}' = \gamma_w \left[\nabla \cdot \mathbf{E} - w\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) + w\mu_0\epsilon_0 \frac{\partial E_z}{\partial t}\right].$$

where, respectively, v and w are the velocities of relative motions in y-direction and z-direction, and γ_v and γ_w are their associated Lorentz factors. Therefore, the **Lorentz transformed Gauss's law of electricity** in arbitrary direction:

$$\nabla' \cdot \mathbf{E}' = \gamma \left[-\mathbf{v} \cdot (\nabla \times \mathbf{B}) + \mu_0\epsilon_0 \frac{\partial}{\partial t}\mathbf{v} \cdot \mathbf{E} + \frac{\rho}{\epsilon_0}\right] \tag{13}$$

where $\mathbf{v}$ is the relative velocity in arbitrary direction, $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$ is the electric charge density and $\gamma = \left(1 - \frac{|\mathbf{v}|^2}{c^2}\right)^{-1}$ is the Lorentz factor in same arbitrary direction as $\mathbf{v}$. Applying the previous calculations on Equation (8) yields the **Lorentz transformed Gauss's law of magnetism** in arbitrary direction:

$$\nabla' \cdot \mathbf{B}' = \gamma \left[\frac{\mathbf{v} \cdot (\nabla \times \mathbf{E})}{c^2} + \mu_0\epsilon_0 \frac{\partial}{\partial t}\mathbf{v} \cdot \mathbf{B} + \rho_m\right] \tag{14}$$

where $\rho_m = \nabla \cdot \mathbf{B}$ is the magnetic charge density. Equations (13) and (14) can be converted into one another (with a negative sign for the curl terms to account for Lenz's law) using the transformation equation $\mathbf{E} = c\mathbf{B}$. The conversion occurs term by term, which leads to the following implication:

$$\nabla \cdot \mathbf{B} = \frac{1}{c} \nabla \cdot \mathbf{E} = \mu_0 c \rho \tag{15}$$

where ρ is the electric charge density. Therefore, this defines the magnetic charge density ρ_m in terms of the electric charge density, as derived from Lorentz transformation of the Maxwell's equation:

$$\rho_m = \mu_0 c \rho \qquad [\mathrm{Wb/m^3}] \tag{16}$$

The magnetic charge density ρ_m has unit of magnetic flux density, [Wb/m^3]. Integrating Equation (16) over a finite volume yields the magnetic monopole, as obtained from the Lorentz transformation of Maxwell's equations:

$$q_m = \mu_0 c q \qquad [\mathrm{Wb}] \tag{17}$$

where q_m is the magnetic charge (monopole), and q is the electric charge. The magnetic monopole has unit of magnetic flux. The weber (symbol: Wb) is the SI unit. Equation (16) enable us to rewrite the four-current vector J^{α} in term of magnetic charge density, that is, $J^{\alpha} = \left(\frac{\rho_m}{\mu_0}, \mathbf{J}\right)$, where $\mathbf{J}$ is the electric current density.

3.2 Lorentz transformation of Maxwell-Ampere's law and Faraday's law

Maxwell-Ampere's law and Faraday's law are central to classical electromagnetism, describing the dynamic relationship between electric and magnetic fields. Faraday's law of induction states that a changing magnetic field induces an electric field. Maxwell-Ampere's law, an extension of Ampère's original law, incorporates the concept of displacement current, accounting for the effects of changing electric fields. This modification by Maxwell was pivotal in predicting the existence of electromagnetic waves, such as light, which propagate at the speed of light in a vacuum. Together, these laws explain phenomena ranging from electromagnetic induction to the behavior of electromagnetic waves. Maxwell-Ampere's law can be obtained in a usual way from Equation (7) by substituting $\beta' = 1,2,3$, assuming the relative uniform motion u between the field inertial frame of reference, S-frame, and the observer inertial frame of reference, S'-frame, is in the x-direction:

$$\frac{\partial}{\partial x^{\alpha'}}F^{\alpha' 1} = \frac{\partial}{\partial x'^0}F'^{01} + \frac{\partial}{\partial x'^1}F'^{11} + \frac{\partial}{\partial x'^2}F'^{21} + \frac{\partial}{\partial x'^3}F'^{31},$$

$$= \left(\frac{\partial B'_{z'}}{\partial y'} - \frac{\partial B'_{y'}}{\partial z'}\right) - \frac{1}{c^2}\frac{\partial E'_{x'}}{\partial t'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}{}_{\alpha}\Lambda^{1}{}_{\beta}F^{\alpha\beta},$$

$$= \gamma_u\left(\frac{\partial}{\partial ct} + \frac{u}{c}\frac{\partial}{\partial x}\right)\Lambda^0{}_\alpha\Lambda^1{}_\beta F^{\alpha\beta} + \gamma_u\left(\frac{u}{c}\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right)\Lambda^1{}_\alpha\Lambda^1{}_\beta F^{\alpha\beta} + \frac{\partial}{\partial y}\Lambda^2{}_\alpha\Lambda^1{}_\beta F^{\alpha\beta} + \frac{\partial}{\partial z}\Lambda^3{}_\alpha\Lambda^1{}_\beta F^{\alpha\beta},$$

which reduces to:

$$\frac{\partial B'_{z'}}{\partial y'} - \frac{\partial B'_{y'}}{\partial z'} = \gamma_u\left[\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - \frac{u\boldsymbol{\nabla}\cdot\mathbf{E}}{c^2} - \frac{1}{c^2}\frac{\partial E_x}{\partial t}\right] + \frac{1}{c^2}\frac{\partial E'_{x'}}{\partial t'}.$$

And

$$\frac{\partial}{\partial x^{\alpha'}}F^{\alpha' 2} = \frac{\partial}{\partial x'^0}F'^{02} + \frac{\partial}{\partial x'^1}F'^{12} + \frac{\partial}{\partial x'^2}F'^{22} + \frac{\partial}{\partial x'^3}F'^{32},$$

$$= \left(\frac{\partial B'_{x'}}{\partial z'} - \frac{\partial B'_{z'}}{\partial x'}\right) - \frac{1}{c^2}\frac{\partial E'_{y'}}{\partial t'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}{}_{\alpha}\Lambda^{2}{}_{\beta}F^{\alpha\beta},$$

$$= \gamma_u\left(\frac{\partial}{\partial ct} + \frac{u}{c}\frac{\partial}{\partial x}\right)\Lambda^0{}_\alpha\Lambda^2{}_\beta F^{\alpha\beta} + \gamma_u\left(\frac{u}{c}\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right)\Lambda^1{}_\alpha\Lambda^2{}_\beta F^{\alpha\beta} + \frac{\partial}{\partial y}\Lambda^2{}_\alpha\Lambda^2{}_\beta F^{\alpha\beta} + \frac{\partial}{\partial z}\Lambda^3{}_\alpha\Lambda^2{}_\beta F^{\alpha\beta},$$

which reduces to:

$$\frac{\partial B'_{x'}}{\partial z'} - \frac{\partial B'_{z'}}{\partial x'} = \left[\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - \frac{1}{c^2}\frac{\partial E_y}{\partial t}\right] + \frac{1}{c^2}\frac{\partial E'_{y'}}{\partial t'}.$$

Similarly

$$\frac{\partial}{\partial x^{\alpha'}}F^{\alpha' 3} = \frac{\partial}{\partial x'^0}F'^{03} + \frac{\partial}{\partial x'^1}F'^{13} + \frac{\partial}{\partial x'^2}F'^{23} + \frac{\partial}{\partial x'^3}F'^{33},$$

$$= \left(\frac{\partial B'_{y'}}{\partial x'} - \frac{\partial B'_{x'}}{\partial y'}\right) - \frac{1}{c^2}\frac{\partial E'_{z'}}{\partial t'} = \frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\Lambda^{\alpha'}{}_{\alpha}\Lambda^{3}{}_{\beta}F^{\alpha\beta},$$

$$= \gamma_u\left(\frac{\partial}{\partial ct} + \frac{u}{c}\frac{\partial}{\partial x}\right)\Lambda^0{}_\alpha\Lambda^3{}_\beta F^{\alpha\beta} + \gamma_u\left(\frac{u}{c}\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right)\Lambda^1{}_\alpha\Lambda^3{}_\beta F^{\alpha\beta} + \frac{\partial}{\partial y}\Lambda^2{}_\alpha\Lambda^3{}_\beta F^{\alpha\beta} + \frac{\partial}{\partial z}\Lambda^3{}_\alpha\Lambda^3{}_\beta F^{\alpha\beta},$$

which reduces to:

$$\frac{\partial B'_{y'}}{\partial x'} - \frac{\partial B'_{x'}}{\partial y'} = \left[\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - \frac{1}{c^2}\frac{\partial E_z}{\partial t}\right] + \frac{1}{c^2}\frac{\partial E'_{z'}}{\partial t'}.$$

By repeating the previous steps for the relative motions in the y-direction and z-direction to identify the general pattern, the following is obtained for the y-direction:

$$\frac{\partial B'_{z'}}{\partial y'} - \frac{\partial B'_{y'}}{\partial z'} = \left[\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - \frac{1}{c^2}\frac{\partial E_x}{\partial t}\right] + \frac{1}{c^2}\frac{\partial E'_{x'}}{\partial t'},$$

$$\frac{\partial B'_{x'}}{\partial z'}-\frac{\partial B'_{z'}}{\partial x'}=\gamma_v\left[\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)-\frac{v\nabla\cdot\mathbf{E}}{c^2}-\frac{1}{c^2}\frac{\partial E_y}{\partial t}\right]+\frac{1}{c^2}\frac{\partial E'_{y'}}{\partial t'},$$

$$\frac{\partial B'_{y'}}{\partial x'}-\frac{\partial B'_{x'}}{\partial y'}=\left[\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-\frac{1}{c^2}\frac{\partial E_z}{\partial t}\right]+\frac{1}{c^2}\frac{\partial E'_{z'}}{\partial t'}.$$

And for z-direction:

$$\frac{\partial B'_{z'}}{\partial y'}-\frac{\partial B'_{y'}}{\partial z'}=\left[\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)-\frac{1}{c^2}\frac{\partial E_x}{\partial t}\right]+\frac{1}{c^2}\frac{\partial E'_{x'}}{\partial t'},$$

$$\frac{\partial B'_{x'}}{\partial z'}-\frac{\partial B'_{z'}}{\partial x'}=\left[\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)-\frac{1}{c^2}\frac{\partial E_y}{\partial t}\right]+\frac{1}{c^2}\frac{\partial E'_{y'}}{\partial t'},$$

$$\frac{\partial B'_{y'}}{\partial x'}-\frac{\partial B'_{x'}}{\partial y'}=\gamma_w\left[\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-\frac{w\nabla\cdot\mathbf{E}}{c^2}-\frac{1}{c^2}\frac{\partial E_z}{\partial t}\right]+\frac{1}{c^2}\frac{\partial E'_{z'}}{\partial t'}.$$

Therefore, the **Lorentz transformed Maxwell-Ampere's law of magnetism**(1):

$$\nabla'\times\mathbf{B}'=\gamma_u\left[(\nabla\times\mathbf{B})_x-\mu_0J_x-\mu_0J_{D,x}\right]\hat{\boldsymbol{x}}+\gamma_v\left[(\nabla\times\mathbf{B})_y-\mu_0J_y-\mu_0J_{D,y}\right]\hat{\boldsymbol{y}}+\gamma_w\left[(\nabla\times\mathbf{B})_z-\mu_0J_z-\mu_0J_{D,z}\right]\hat{\boldsymbol{z}} \tag{18}$$

where J_x, J_y, J_z are the electric currents densities in x, y, z-directions respectively, and $J_{D,x}, J_{D,y}, J_{D,z}$ are the electric displacement currents densities in x, y, z-directions respectively. Applying the previous calculations on Equation (8) yields the **Lorentz transformed Faraday's law for induction**(2):

$$\nabla'\times\mathbf{E}'=\gamma_u\left[(\nabla\times\mathbf{E})_x+J_{m,x}+J_{B,x}\right]\hat{\boldsymbol{x}}+\gamma_v\left[(\nabla\times\mathbf{E})_y+J_{m,y}+J_{B,y}\right]\hat{\boldsymbol{y}}+\gamma_w\left[(\nabla\times\mathbf{E})_z+J_{m,z}+J_{B,z}\right]\hat{\boldsymbol{z}} \tag{19}$$

where $J_{m,x}, J_{m,y}, J_{m,z}$ are the magnetic currents densities in x, y, z-directions, and the quantities; $J_{B,x}=\partial B_x/\partial t, J_{B,y}=\partial B_y/\partial t, J_{B,z}=\partial B_z/\partial t$, are the magnetic displacement currents densities in the same mentioned directions. Again, Equations (18) and (19) can be transformed into one another term by term by applying the transformation equation $\mathbf{E}=c\mathbf{B}$. However, except for the curl terms, the signs of the current densities and displacement current densities reverse, requiring the use of $\mathbf{E}=-c\mathbf{B}$ instead.

4. Lorentz transformation of electromagnetic stress–energy tensor

The electromagnetic stress–energy tensor is a cornerstone of classical electromagnetism, describing the distribution and transfer of energy, momentum, and stress within electromagnetic fields. It captures the energy density of the fields, the flow of energy through space, and the forces exerted on matter. A closely related concept is the Poynting vector, which quantifies the direction and magnitude of energy flow carried by electromagnetic fields, offering an intuitive representation of energy transfer within a system. These concepts are foundational for understanding how electromagnetic energy propagates and interacts with its surroundings. In addition to describing energy transfer, the electromagnetic stress–energy tensor ensures the conservation of energy and momentum within electromagnetic fields. These conservation laws reflect the fundamental physical principles that energy and momentum cannot be created or destroyed but only transferred between the fields and matter. The Lorentz force and Lorentz force density further detail how these fields act on charged particles and currents, exerting forces that drive motion or generate stresses. Together, these principles provide a unified framework for analyzing the dynamics of electromagnetic fields and their interactions with matter, making the tensor an indispensable tool in both theoretical physics and practical applications.

Lorentz transformation of electromagnetic stress–energy tensor can be obtained by expanding the tensor equation (20) over index $\beta'=0,1,2,3$, and in x, y, z-directions for every index.

$$\frac{\partial}{\partial x^{\alpha'}}\boldsymbol{T}^{\alpha'\beta'}=\frac{\partial x^{\delta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\delta}}\boldsymbol{\Lambda}^{\alpha'}{}_{\alpha}\boldsymbol{\Lambda}^{\beta'}{}_{\beta}\boldsymbol{T}^{\alpha\beta}, \tag{20}$$

The process yields the two following transformed equations(3):

(1) The contribution of displacement current $1/c^2(\partial\mathbf{E}'/\partial t')$ has been set to zero.

(2) The contribution of displacement current $1/c^2(\partial\mathbf{B}'/\partial t')$ has been set to zero.

(3) The contribution of electromagnetic energy density $\partial\rho'_{EM}/\partial t'$ has been set to zero.

$$\boldsymbol{\nabla}' \cdot \mathbf{S}' = \boldsymbol{\gamma}\left[\boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{f} \cdot \mathbf{v} + \frac{\partial \rho_{EM}}{\partial t}\right] \tag{21}$$

And

$$\mathbf{f}' = \gamma_u\left[f_x + \frac{u}{c^2}\left(\frac{\partial \rho_{EM}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S}\right)\right]\hat{\boldsymbol{x}} + \gamma_v\left[f_y + \frac{v}{c^2}\left(\frac{\partial \rho_{EM}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S}\right)\right]\hat{\boldsymbol{y}} + \gamma_w\left[f_z + \frac{w}{c^2}\left(\frac{\partial \rho_{EM}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S}\right)\right]\hat{\boldsymbol{z}} \tag{22}$$

where $\mathbf{S}$ is the Poynting vector, ρ_{EM} the electromagnetic energy density, and $\mathbf{f}$ the Lorentz force density. Therefore, the law of **Lorentz transformed electromagnetic Lorentz force**:

$$\mathbf{F}' = \gamma_u\left[F_x + \frac{u}{c^2}\frac{d}{dt}\int_V \rho_{EM}\, d^3x + \frac{u}{c^2}\int_S \mathbf{S} \cdot d\mathbf{a}\right]\hat{\boldsymbol{x}} + \gamma_v\left[F_y + \frac{v}{c^2}\frac{d}{dt}\int_V \rho_{EM}\, d^3x + \frac{v}{c^2}\int_S \mathbf{S} \cdot d\mathbf{a}\right]\hat{\boldsymbol{y}} + \gamma_w\left[F_z + \frac{w}{c^2}\frac{d}{dt}\int_V \rho_{EM}\, d^3x + \frac{w}{c^2}\int_S \mathbf{S} \cdot d\mathbf{a}\right]\hat{\boldsymbol{z}}$$

5. Lorentz transformed Maxwell's equation at non-relativistic velocities

In the non-relativistic limit, where $|\mathbf{v}| \ll c$, the binomial expansion can be applied to expand γ and approximate Equations (13), (14), (18), and (19). By neglecting higher-order terms in the expansion and terms involving the factor $|\mathbf{v}|^2/c^2$—since they are divided by c^2 and c^4, which are extremely large—we obtain:

$$\begin{aligned}
\boldsymbol{\nabla}' \cdot \mathbf{E}' &= -\mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) + \mu_0\epsilon_0 \frac{\partial}{\partial t}\mathbf{v} \cdot \mathbf{E} + \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla}' \cdot \mathbf{B}' &= \frac{\mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{E})}{c^2} + \mu_0\epsilon_0 \frac{\partial}{\partial t}\mathbf{v} \cdot \mathbf{B} + \rho_m \\
\boldsymbol{\nabla}' \times \mathbf{E}' &= \boldsymbol{\nabla} \times \mathbf{E} + \mathbf{J_m} + \mathbf{J_B} \\
\boldsymbol{\nabla}' \times \mathbf{B}' &= \boldsymbol{\nabla} \times \mathbf{B} - \mu_0\mathbf{J} - \mu_0\mathbf{J_D}
\end{aligned} \tag{23}$$

Equations (21) and (22) upon binomial approximation become:

$$\boldsymbol{\nabla}' \cdot \mathbf{S}' = \boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{f} \cdot \mathbf{v} + \frac{\partial \rho_{EM}}{\partial t}, \qquad f' = f + \frac{v}{c^2}\frac{\partial \rho_{EM}}{\partial t} + \frac{v}{c^2}\boldsymbol{\nabla} \cdot \mathbf{S} \tag{24}$$

And the Lorentz force:

$$\mathbf{F}' = \mathbf{F} + \frac{v}{c^2}\frac{d}{dt}\int_V \rho_{EM}\, d^3x + \frac{v}{c^2}\int_S \mathbf{S} \cdot d\mathbf{a}$$

Although Equations (13), (14), (18), and (19) represent Lorentz-transformed Maxwell's equations, they could also be referred to as the "Theory of Relativistic Dyno-electromagnetism." This is because the scalar fields—$\boldsymbol{\nabla}' \cdot \mathbf{E}'(\mathbf{x}, \dot{\mathbf{x}}, t), \boldsymbol{\nabla}' \cdot \mathbf{B}'(\mathbf{x}, \dot{\mathbf{x}}, t)$—and the vector fields—$\boldsymbol{\nabla}' \times \mathbf{E}'(\mathbf{x}, \dot{\mathbf{x}}, t), \boldsymbol{\nabla}' \times \mathbf{B}'(\mathbf{x}, \dot{\mathbf{x}}, t)$—are velocity-dependent and include relativistic corrections. On the other hand, Equations (23) might be labeled the "Theory of Dyno-electromagnetism." Dyno-electromagnetism is a new theoretical framework that extends Maxwell's equations by applying the Lorentz transformation, thereby integrating relativistic mechanics and introducing velocity dependence alongside position and time. The term "dyno" is derived from the Greek word "dynamis" (δύναμις), meaning "power" or "force," reflecting the framework's focus on the dynamic relationship between motion, energy, and electromagnetic fields. In this framework, the divergence and curl of the electric **E** and magnetic **B** fields are generalized to incorporate the effects of relative motion or velocity, offering a more comprehensive understanding of electromagnetic phenomena in dynamic systems.

6. A model of monopoles from Lorentz-transformed Gauss's law of magnetism

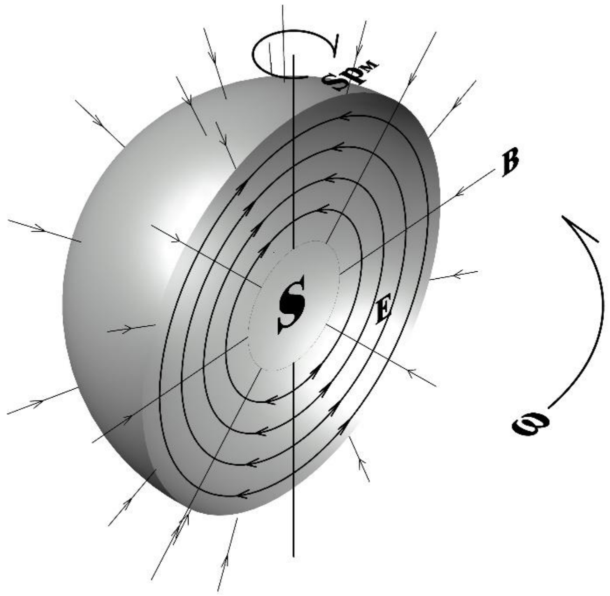

Figure (1): Cutting through the monopole exposes its internal composition and illustrates the connection between E, B, and ω

Lorentz-transformed Gauss's law of magnetism, in its formulation, offers a potential mechanism that could be involved in the creation of magnetic monopoles, particularly through the term $\gamma(\mathbf{v}\cdot(\nabla\times\mathbf{E})/c^2)=\nabla'\cdot\mathbf{B}'$. To properly investigate this proposed model of a magnetic monopole, it is necessary to express this term in spherical coordinates. Hence, expanding Equation (8) in spherical coordinates requires utilizing Lorentz transformation and the dual electromagnetic field tensor, both expressed in their corresponding spherical coordinate forms [(4)]. The following assumptions are made: the electric field $\mathbf{E}$ is oriented in the $-\theta$-direction[(5)], representing a uniform electric field in the $-\theta$-direction, emanates from the bottom of a sphere, and converges at the top (see Figure (1)), $\mathbf{E}=(0,\ -E_\theta,0)$, in the monopole frame of reference (S); the relative uniform motion $\mathbf{v}=(0,0,v_\phi)$ is characterized by the tangential velocity v_ϕ in the $+\phi$-direction, where $v_\phi = r\sin\theta\,\omega = v'_{\phi'}$ ($\omega = \dot{\phi}$, the angular frequency of the electric field (ccw)); the induced magnetic field $\mathbf{B}'$ lies in the radial $-r'$-direction, converging toward the center of the sphere, $\mathbf{B}'=(-B'_{r'},0,0)$, in the observer's frame of reference (S')[(6)]. Additionally, E_θ is constant, and $B'_{r'}$ is spherically symmetrical with respect to θ' and ϕ', varying only with the radial coordinate r' and remaining independent of time t', as E_θ and v_ϕ are constant. While this will provide the full derivation of the proposed monopole model, including relativistic corrections, nevertheless, for the purposes of this paper, Equation (23) will be used instead. Consequently, we obtain:

$$\iint \mathbf{B}'\cdot d\mathbf{a}' = -\frac{\omega E_\theta}{c^2}\iiint r^2\sin^2\theta\, dr d\theta d\phi \tag{25}$$

Integrating over the volume of the particle's sphere and applying the divergence theorem to the left-hand side, and one can utilize the facts that $r=r'$, $\theta=\theta'$, and $d\phi=d\phi'$ in the Galilean transformation of rotating frames (Mazumdar & Parida, 2020) (the result expresses both sides of the equation using prime notation, but this notation has been removed for simplicity):

$$\iint \mathbf{B}\cdot d\mathbf{a} = \mathrm{Q_M} = -\frac{\omega E_\theta}{c^2}\iiint r^2\sin^2\theta\, dr d\theta d\phi$$

---

(4) To ensure a comprehensive understanding of the presented model, it is advisable to refer to the approximate illustration of "*Electron spin and magnetic moment*" by Purcell (2013).

(5) The convention used for θ is that it represents the *inclination* angle.

(6) For the right hand rule used in case of moving electromagnetic field, referes to "*Electrodynamics and Relativity*" by Griffiths (2023).

Hence, the magnetic monopole Q_M is:

$$Q_M = -\frac{\pi^2}{3c^2}\omega R^3 E_\theta \qquad [\mathrm{Wb}] \tag{26}$$

where R represents the radius of the monopole, and the presence of the negative sign indicates that the monopole carries a charge of S-pole. The electromagnetic momentum density field inside the monopole is given by:

$$\mathbf{p}_{\mathrm{em}} = -\,\epsilon_0 E_\theta \hat{\boldsymbol{\theta}} \times -B_r \hat{\mathbf{r}} \tag{27}$$

which point into $-\phi$-direction (counter to $\mathbf{v}$). This field curls over the entire interior of the spherical volume of the monopole particle(7). The formula $|\mathbf{p}_{\mathrm{em}}|/c = \rho_{mass}$ is suggested to provide what could be referred to as the electromagnetic mass density. Hence, the mass M_{em} and the spin Sp_M of the monopole are:

$$M_{\mathrm{em}} = \iiint \rho_{mass}\ r^2 \sin\theta\ dr d\theta d\phi \quad \text{and} \quad \mathrm{Sp}_M = \iiint r\sin\theta\,|\mathbf{p}_{\mathrm{em}}|\ r^2 \sin\theta\ dr d\theta d\phi \tag{28}$$

The same logic used to derive formulas (26) to (28) can be applied to derive similar formulas for electric monopoles, utilizing the equation $\nabla'\cdot\mathbf{E}' = -\,\mathbf{v}\cdot(\nabla\times\mathbf{B})$ in this case. Furthermore, the combination of $\mathbf{E} = (0,\ E_\theta, 0)$ and $\mathbf{v} = (0, 0, r\sin\theta\,(-\omega))$ equally materializes a monopole with S-charge. However, this differs from the first case by exhibiting a reversed spin direction (clockwise). The following table outlines the key characteristics of electric and magnetic monopoles.

Table (1): Key characteristics of electric and magnetic monopoles

Orientations of the field and frequency	Orientation of the spin	Type of the monopole
$(\boldsymbol{E_\theta}, \boldsymbol{\omega})$	Spin-down	N-charge magnetic monopole
$(-\boldsymbol{E_\theta}, -\boldsymbol{\omega})$	Spin-up	N-charge magnetic monopole
$(-\boldsymbol{E_\theta}, \boldsymbol{\omega})$	Spin-down	S-charge magnetic monopole
$(\boldsymbol{E_\theta}, -\boldsymbol{\omega})$	Spin-up	S-charge magnetic monopole
$(\boldsymbol{B_\theta}, \boldsymbol{\omega})$	Spin-down	$(-)$ charge electric monopole
$(-\boldsymbol{B_\theta}, -\boldsymbol{\omega})$	Spin-up	$(-)$ charge electric monopole
$(-\boldsymbol{B_\theta}, \boldsymbol{\omega})$	Spin-down	$(+)$ charge electric monopole
$(\boldsymbol{B_\theta}, -\boldsymbol{\omega})$	Spin-up	$(+)$ charge electric monopole

7. Discussion

An analysis of the monopole model reveals that an observer in the particle's rest frame (S-frame) detects no charge, mass, or spin. However, a second observer (S'-frame), relative to whom the S-frame is rotating counterclockwise (ccw), will observe the particle to possess charge, mass, and clockwise (cw) spin. Notably, a third observer (S''-frame), rotating ccw relative to the S'-frame but with a higher relative velocity than the S-frame, will observe the particle with mass (potentially different from that observed by the S'-frame, depending on the relative velocities difference), an opposite charge relative to the S'-frame observation, and ccw spin.

It is assumed that the monopole model must be constrained to maintain a 90-degree angle between the dipole-like field (whether electric or magnetic, e.g., E_θ) and the tangential relative velocity, v_ϕ. This dipole field and the tangential relative velocity are also permitted to undergo rotational transformation only through 180 degrees(8) (a reflection transformation) to preserve their perpendicularity. These constraints ensure alignment between the classical monopole models, derived from Lorentz transformed Gauss's laws of electricity and magnetism, and the quantum theoretical framework, as the angle in the dot product $\mathbf{v}\cdot(\nabla\times\mathbf{E})$ or $\mathbf{v}\cdot(\nabla\times\mathbf{B})$

(7) Worth mentioning $\nabla\times\mathbf{p}_{\mathrm{em}}$ itself is a vector field presents in every electric or magnetic charged particle or antiparticle. This field seems to sustain particle mass by continuously flowing in and out of the particle's volume. It may extend and permeate throughout the entire universe.

(8) It has been assumed that transitions between polarization states ($E_\theta \leftrightarrow -E_\theta$ or $B_\theta \leftrightarrow -B_\theta$) through reflection transformations are possible. Consequently, the charged particle (whether electric or magnetic) will switch its charge polarity if such a transition occurs, while the particle's spin direction remains constant.

could otherwise take any arbitrary value in classical formulations. This restriction further implies that the magnetic monopole belongs to the fermion particle family, specifically exhibiting spin-1/2 (up and down) properties. Based on these assumptions, Table (1) has been constructed.

Without loss of generality, the **S**-charge magnetic monopole (above) can be understood as a unique electromagnetic wave—a localized wave rotating around a central axis and bounded within a defined volume. As is typical, the electric field ($\mathbf{E}$), magnetic field ($\mathbf{B}$), and the Poynting vector ($\mathbf{S}$) remain mutually perpendicular to each other.

By representing a charged particle as a spinning, localized electromagnetic wave—initiated by the spin of a microscopic dipole (either electric or magnetic), which is formed through the alignment of the electric field $\mathbf{E}$ or magnetic field $\mathbf{B}$ along the θ-direction in the spherical coordinate system—and integrating this with the findings from Table (1) (which show that a charged particle exhibits two spin states), further insights can be drawn. Specifically, when the tangential relative velocity of the field remains in the $+\phi$-direction (particle spins down), the magnetic field $\mathbf{B}$ displays two polarization states: one aligned vertically upward along the $-\theta$-direction, $(-B_\theta, \omega)$, corresponding to the (+)-charged particle, and the other aligned vertically downward along the $+\theta$-direction, (B_θ, ω), corresponding to the (−)-charged particle. Similarly, the electric field $\mathbf{E}$ exhibits the same two polarization states: vertically upward along the $-\theta$-direction, $(-E_\theta, \omega)$, for the **S**-charged particle, and vertically downward along the $+\theta$-direction, (E_θ, ω), for the **N**-charged particle (with the tangential relative velocity also remaining in the $+\phi$-direction). The same observations of the two polarization states associated with the $\mathbf{E}$ and $\mathbf{B}$ fields apply to the case of tangential relative velocity in the $-\phi$-direction (particle spins up), but with an alternation in the polarity of the corresponding charges. Simultaneously, each charge polarity ((+), (−), **N**, **S**) encompasses, in its characteristics, the two field polarization states and the two particle spin states. Also, the electric charge polarities (+) and (−) share the two field polarization states and the two spin states, as do the magnetic charge polarities **N** and **S** (each in relation to their respective fields). Moreover, the four combinations of the two field polarization states and the two spin states cover all possible manifestations of charge polarities. Therefore, the polarity of a charged particle (whether electric or magnetic) can be considered a discrete **state** rather than an intrinsic **constant** property. This is because, as shown above, it depends on discrete states of polarization and spin[(9)].

Given the quantum principle that the spin state of a charged particle is a superposition of its two spin states (up and down), and considering the above reasoning regarding discrete states of charge polarity, it is reasonable now to extend this idea to the charge of the particle, suggesting that the charge itself exists in a superposition of two charge states: (+) and (−) for an electric particle, and **N** and **S** for a magnetic particle—this reflects the superposition of its dipole field polarization states[(10)] (vertically up and down) per spin state. Since a charged particle can exist in any of these charge states, the term "unicharge" (derived from "unified charge," which refers to a single particle exhibiting both charge polarities, (+) and (−) or **N** and **S**) can aptly describe this superposition state, whether the particle is electric or magnetic in nature.

The electric unicharge particles which embody a superposition of (+) and (−) charge states as well as up and down spin states, are governed by **Dirac's equation**. While Dirac's equation typically treats electrically charged particles and their antiparticles as distinct entities, the electric unicharge concept redefines Dirac's spinor to accommodate unicharge states: (−) and (+) charge states with their corresponding up and down spin states. It now accommodates four states of a single particle, unifying the particle and its antiparticle into one entity. In principle, the spinor components accommodating the particle, and its antiparticle have been renamed as the (−) and (+) charge states, respectively. The particle's spin states indirectly characterize the spin states of the dipole (since the dipole's spin direction is opposite to that of the particle). A prominent example of an electric unicharged particle is the electron-positron pair, which, owing to its importance, can be termed the "unitron." This naming scheme can be generalized to other charged particles, such as referring to the proton-antiproton pair as the "uniproton," and so on. Dirac's equation can also be adapted to describe magnetic unicharged particles by reinterpreting the (−)-charge state as the **S**-charge state and the (+)-charge state as the **N**-charge state.

Given that the radius of the magnetic monopole is expected to be extremely small, as with any particle, the ratio R^3/c^2 would result in a very small magnitude of charge, making it potentially difficult to detect.

The conventional derivation of electromagnetic waves now requires not only that the equations be source-free but also that they be expressed in a proper frame of reference (the relative velocity is required to be set to zero).

While the modified framework provides a reasonable explanation for how particles or antiparticles, whether magnetically or electrically charged, come into existence, it does not explain why nature assigns specific values to certain characteristics of these particles, such as mass, charge, magnetic moment, and others. This limitation could potentially be addressed if the framework is successfully quantized, or at least if the model of particle formation is quantized.

(9) Polarity has also been previously demonstrated to be a relativistic frame-dependent quantity in the context of continuous change.

(10) They represent a new internal degree of freedom that characterizes charged particles, in addition to their spin degree of freedom.

Additionally, the framework does not directly address chargeless and massless particles in the same way it does for charged particles. While the assumption of two counter-rotating dipoles might provide a potential solution, it appears arbitrary and would require experimental validation to confirm its validity. Nevertheless, the mathematical steps—starting with writing the tensor equation in a specific frame of reference, substituting the tensors on the left-hand side with their transformed expressions, and expanding the equation—can be applied to other relevant scenarios in physics to reveal deeper structural insights.

8. Concluding remarks

It is shown that both the magnetic charge density and magnetic current density can be derived and seamlessly integrated into Maxwell's equations via a direct and rigorous approach. The methodology is deemed direct, as it originates from the electromagnetic field tensor without reliance on additional assumptions, and safer, as it is accomplished solely through mathematical deduction.

In the deduced framework, Maxwell's equations are redefined as transformation equations, establishing a framework in which the divergence and curl of electric and magnetic fields become frame-dependent quantities that vary under Lorentz transformations. These quantities are demonstrated to depend not only on space and time but also on the uniform relative velocity between the electromagnetic field's reference frame and the observer's reference frame. Consequently, electromagnetic induction, in all its aspects, is revealed to be a relativistic phenomenon. Furthermore, it is shown that an electric charge can originate purely from a magnetic source, and a magnetic charge can originate purely from an electric source.

One of the most significant findings is the theoretical demonstration of the non-vanishing divergence of the magnetic field, meaning $\nabla \cdot \mathbf{B} \neq 0$. It is shown that the divergence of magnetic field does not always equal zero, contrary to Gauss's law of magnetism. This result indicates that the existence of a magnetic monopole can arise as a consequence of the relative motion of an electric field. It underscores that the observed absence of magnetic charge density (ρ_m) and magnetic field ($\mathbf{B}$) in one inertial frame does not hold universally across all reference frames. Instead, the transformation between reference frames can give rise to magnetic phenomena, even if they were absent in the original frame, due to the relativistic relationship between electric and magnetic fields.

It is also shown that the restoration of magnetic charge density and magnetic current density to Maxwell's equations achieves a remarkable symmetry between electric and magnetic fields, particularly through their interconversion under the relation $\mathbf{E} = c\mathbf{B}$. This symmetry may reduce Maxwell's equations to just two equations—one for the divergence of the field and another for its curl. By introducing the change of variables $\mathbf{E} = c\mathbf{B}$ and $\mathbf{E}' = c\mathbf{B}'$, these two equations can be written equivalently for either the electric or magnetic field. This elegant unification not only enhances the theoretical framework of electromagnetism but also underscores the profound beauty and coherence inherent in the laws governing electromagnetic phenomena. Furthermore, this highlights another triumph of the theory of relativity, as its application to Maxwell's equations restores all missing terms, elevating electromagnetism to its full glory—a success that may stand on equal footing with the discovery of gravitational waves and black holes.

It is also shown that the proposed monopole model exhibits a remarkable dependence on the observer's frame of reference, underscoring the fundamental influence of relativity on its physical characteristics and revealing insights into the nature of charge, mass, and spin. Specifically, in the particle's rest frame (S-frame), the absence of charge, mass, and spin is demonstrated, highlighting the intrinsic neutrality of the system in its own reference state. However, when observed from a rotating frame (S'-frame) moving clockwise relative to the S-frame, the particle manifests charge, mass, and clockwise spin, emphasizing the role of relative motion in the emergence of these properties. Further complexity is unveiled in the S''-frame, where an observer rotating counterclockwise relative to the S'-frame but with a greater relative velocity than the S-frame detects a mass potentially distinct from that observed in the S'-frame, an inverted charge, and counterclockwise spin. These findings illustrate the intricate interplay between relative motion and the observed physical properties, emphasizing the frame-dependent nature of fundamental quantities in the monopole model. Hence, it can be inferred that tiny dipoles might exist, which could act as compelling candidates for boson particles responsible for granting mass, charge, and spin to other fundamental particles.

Although physics literature describes charge, mass, and spin as intrinsic properties of charged particles, it has been demonstrated that they are not intrinsic but rather relative phenomena. Furthermore, while charge is traditionally considered an intrinsic property, leading to the association of dipole materialization with spin charge, it has been shown that the dipole is likely the most fundamental entity, responsible for the emergence of all these properties.

Emerging from microscopic dipole field dynamics, this framework establishes spin orientation and polarization configuration as interdependent quantum degrees of freedom that collectively define charge manifestation. The results in Table (1) reveal how this dual relationship naturally gives rise to the unicharge paradigm, where particles exist as quantum superpositions of both charge states (+/− or N/S) and spin orientations. This perspective renders the conventional particle-antiparticle distinction obsolete, instead presenting matter and antimatter as different quantum configurations of the same fundamental entities through their spin-polarization entanglement. Remarkably, Dirac's equation intrinsically incorporates this unified view, naturally describing both electric and magnetic unicharge systems via their coupled spin-polarization dynamics.

The framework further demonstrates that reflection symmetry induces correlated transformations between spin and polarization states, establishing charge polarity as an emergent property of their dynamic interplay (The wave nature of charged particles fundamentally links spin and polarization states in determining charge polarity). By revealing how charge, spin, and polarization form an inseparable triad of wave properties, this approach provides a unified foundation connecting classical and quantum physics through the intrinsic behavior of localized electromagnetic waves.

It is also shown that Maxwell's original equations are valid only at non-relativistic velocities and that they are not an exact theory but an approximation to a more general relativistic theory. Additionally, it is demonstrated that the electric and magnetic charge densities (and hence the charges) are scalar fields that vary with position, change in position, and time.

References:

- Dirac, P. A. M. (1931). *Quantised singularities in the electromagnetic field*. Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, **133**(**821**), 60-72.
- Dirac, P. A. M. (1948). *The theory of magnetic poles*. Physical Review, **74**(**7**), 817.
- t Hooft, G. (1974). *Magnetic monopoles in unified theories*. Nucl. Phys. B, **79**(**CERN-TH-1876**), 276-284.
- Polyakov, A.M. (1996). *Particle Spectrum in Quantum Field Theory* - JETP Lett. **20**, 194 (1974).
- Prasad, M. K., & Sommerfield, C. M. (1975). *Exact classical solution for the't Hooft monopole and the Julia-Zee Dyon*. Physical Review Letters, **35**(**12**), 760.
- Seiberg, N., & Witten, E. (1994). *Electric-magnetic duality, monopole condensation, and confinement in N= 2 supersymmetric Yang-Mills theory*. Nuclear Physics B, **426**(**1**), 19-52.
- Konishi, K. (2007). *The magnetic monopoles seventy-five years later*. In String Theory and Fundamental Interactions: Gabriele Veneziano and Theoretical Physics: Historical and Contemporary Perspectives (pp. 471-521). Berlin, Heidelberg: Springer Berlin Heidelberg.
- Preskill, J. P. (1979). *Cosmological production of superheavy magnetic monopoles*. Physical Review Letters, **43**(**19**), 1365.
- Guth, A. H. (1981). *Inflationary universe: A possible solution to the horizon and flatness problems*. Physical Review D, **23**(**2**), 347.
- Castelnovo, C., Moessner, R., & Sondhi, S. L. (2008). *Magnetic monopoles in spin ice*. Nature, **451**(**7174**), 42-45.
- Morris, D. J. P., Tennant, D. A., Grigera, S. A., Klemke, B., Castelnovo, C., Moessner, R., ... & Perry, R. S. (2009). *Dirac strings and magnetic monopoles in the spin ice Dy2Ti2O7*. Science, **326**(**5951**), 411-414.
- Giblin, S. R., Bramwell, S. T., Holdsworth, P. C., Prabhakaran, D., & Terry, I. (2011). *Creation and measurement of long-lived magnetic monopole currents in spin ice*. Nature Physics, **7**(**3**), 252-258.
- Ray, M. W., Ruokokoski, E., Kandel, S., Möttönen, M., & Hall, D. S. (2014). *Observation of Dirac monopoles in a synthetic magnetic field*. Nature, **505**(**7485**), 657-660.
- Cabrera, B. (1982). *First results from a superconductive detector for moving magnetic monopoles*. Physical Review Letters, **48**(**20**), 1378.
- MACRO collaboration, & Ambrosio, M. (2002). *Final results of magnetic monopole searches with the MACRO experiment*. The European Physical Journal C-Particles and Fields, **25**(**4**), 511-522.
- Fagaly, R. L. (2006). *Superconducting quantum interference device instruments and applications*. Review of scientific instruments, **77**(**10**).
- Mavromatos, N. E., & Mitsou, V. A. (2020). *Magnetic monopoles revisited: Models and searches at colliders and in the Cosmos*. International Journal of Modern Physics A, **35**(**23**), 2030012.
- Kirschner, F. K., Flicker, F., Yacoby, A., Yao, N. Y., & Blundell, S. J. (2018). *Proposal for the detection of magnetic monopoles in spin ice via nanoscale magnetometry*. Physical Review B, **97**(**14**), 140402.
- Lorentz, H. A. (1937). *Electromagnetic phenomena in a system moving with any velocity smaller than that of light*. In Collected Papers: Volume **V** (pp. 172-197). Dordrecht: Springer Netherlands.
- Michelson, A. A., & Morley, E. W. (1887). *On the relative motion of the Earth and the luminiferous ether*. American journal of science, **3(203)**, 333-345.
- Maxwell, J. C. (1865). VIII. *A dynamical theory of the electromagnetic field*. Philosophical transactions of the Royal Society of London, (**155**), 459-512.
- Einstein, A. (1905). *On the electrodynamics of moving bodies*.
- Purcell, E. M. (2013). *Electricity and magnetism*. Cambridge university press.
- Griffiths, D. J. (2023). *Introduction to electrodynamics*. Cambridge University Press.
- Mazumdar, S., & Parida, K. K. (2020). *Relativistic Transformation of Spherical Coordinates* (t, r, θ, ϕ). arXiv preprint arXiv:2008.08780.