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A class of analytic functions and it`s properties

Salma Faraj Ramadan

Faculty of Science || Sabratha University || Libya

Abstract: (Ramadan & Darus, 2019, p37) recently introduced a new class of analytic functions $Q(\mu,\varepsilon),\ \varepsilon,\mu\in[0,1)$ in the complex plane £ . In this present paper we give sufficient conditions for the functions belonging to this class, coefficient inequalities, Growth and distortion theorem and closure theorem are obtained.

Keywords: Analytic Functions - Properties of Analytic Functions.

فئة من الدوال التحليلية وخواصها

سالمة فرج رمضان

كلية العلوم || جامعة صبراتة || ليبيا

الملخص: مؤخرا (في سنة 2019، الدكتورة/ سالمة رمضان والبروفيسور/ ماسلينا داروس) قدمتا فئة جديدة من الدوال التحليلية والتي يرمز لها بالرمز $Q(\mu,\varepsilon),\ \varepsilon,\mu\in[0,1)$ في المستوى المركب £. في هذه الورقة سنعطي الشروط الكافية للدوال التحليلية لكي تتبع هذه الفئة، وسندرس متباينات المعاملات لهذه الدوال، أيضا نظرية النمو والتشوه كذلك نظرية الاغلاق لهذه الفئة من الدوال.

الكلمات المفتاحية: الدوال التحليلية - خواص الدوال التحليلية.

1. Introduction

Let A denote a class of all analytic functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $U=\{z\in £ : |z|<1\}$ in the complex plane £ and normalized by $f(0)=f'(0)-1=0$. We denote by S the class of all functions in A which are univalent in U . Next, we state basic ideas on the familiar subclasses of A consisting of functions that are starlike of order $\alpha\left(\alpha\in[0,1)\right)$ in U , convex of order $\alpha\left(\alpha\in[0,1)\right)$ in U . By definition, we have a function belonging to A is said to be starlike of order $\alpha\left(\alpha\in[0,1)\right)$ in U if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha, \qquad z\in U$$

We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order $\alpha(\alpha \in [0,1))$ in U . Also, a function belonging to A is said to be convex of order $\alpha(\alpha \in [0,1))$ in U if it satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in U$$

We denote by $C(\alpha)$ the subclass of A consisting of functions which are convex of order $\alpha(\alpha \in [0,1))$ in U . (see [3], [6]).

Interesting generalization of the functions classes $S^*(\alpha)$ and $C(\alpha)$ was introduced and studied by a lot of authors, (for details, one can refer to [1], [5]).

The propose of the present paper is to study various properties for functions belonging to a new class $Q(\mu,\varepsilon)$ which introduced by Salma F. Ramadan and M. Darus [4] as follows

Definition1.1: A function f given by (1.1) is said to be in the class $Q(\mu,\varepsilon)$ if the following condition is satisfied

$$Q(\mu,\varepsilon)=\left\{f \in A : \mathrm{Re}\left(\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}\right) > \varepsilon,\ \varepsilon,\mu \in [0,1)\right\} \qquad (1.2)$$

It is clear that the condition (1.2) holds true if

$$\left|\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}-1\right| \le 1-\varepsilon .$$

In order to proof our next theorem, we have to recall the following lemma

Lemma1.2: see [2] Let $w(z)$ be analytic in U and such that $w(0)=0$.Then if $|w(z)|$ attains its maximum value on circle $|z|=r<1$ at a point $z_0 \in U$,we have

$$z_0 w'(z_0)=kw(z_0),$$

where $k \ge 1$ is a real number.

2. Condition for the class $Q(\mu,\varepsilon)$

Theorem 2.1: If $f \in A$, then the function $f(z)$ belongs to the class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$ if the following condition is satisfied

$$\left|\frac{1+z\left(zf'(z)\right)''}{1+f''(z)}-\frac{z^2(1-\mu)f''(z)+zf'(z)}{\mu f(z)+z(1-\mu)f'(z)}\right| < \frac{1-\varepsilon}{2-\varepsilon}. \qquad (2.1)$$

Proof: Assume $f \notin Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$, so by the definition 1.1

$$\left|\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}-1\right| \ge 1-\varepsilon ,$$

we define the function $w(z)$ by

$$\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}=1+(1-\varepsilon)w(z) \tag{2.2}$$

Then $w(z)$ is analytic in U and $w(0)=0$. By logarithmic differentiations, we get from (2.2) that

$$\frac{1+z\left(zf'(z)\right)''}{1+f''(z)}-\frac{z^2(1-\mu)f''(z)+zf'(z)}{\mu f(z)+z(1-\mu)f'(z)}=\frac{(1-\varepsilon)zw'(z)}{1+(1-\varepsilon)w(z)} \tag{2.3}$$

Suppose their exist $z_0\in U$ such that

$$\max|w(z)|=|w(z_0)|=1,\quad |z|<|z_0|,$$

then from lemma 1.2, we have

$$z_0w'(z_0)=kw(z_0),\quad k\geq 1.$$

Letting $w(z_0)=e^{i\theta}$, and substitution z_0 in (2.3), we have

$$\left|\frac{1+z_0\left(z_0f'(z_0)\right)''}{1+f''(z_0)}-\frac{z_0^2(1-\mu)f''(z_0)+zf'(z_0)}{\mu f(z_0)+z_0(1-\mu)f'(z_0)}\right|=\left|\frac{(1-\varepsilon)z_0w'(z_0)}{1+(1-\varepsilon)w(z_0)}\right|$$

$$=\left|\frac{(1-\varepsilon)ke^{i\theta}}{1+(1-\varepsilon)e^{i\theta}}\right|$$

$$\geq\frac{(1-\varepsilon)k\left|e^{i\theta}\right|}{1+(1-\varepsilon)\left|e^{i\theta}\right|}$$

$$\geq\frac{1-\varepsilon}{2-\varepsilon}.$$

Which contradicts our assumption (2.1). There for $|w(z)|<1$ holds for all $z\in U$. Ultimately, from (2.2) we have

$$\left|\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}-1\right|=(1-\varepsilon)|w(z)|<1-\varepsilon.$$

That is, $f\in Q(\mu,\varepsilon),\ \varepsilon,\mu\in[0,1)$. Thus the proof of Theorem 2.1 completed.

Taking $\varepsilon=0$ and $f(z)$ giving by (1.1), we have the following corollary

Corollary 2.2: If $f\in A$ given by (1.1) and satisfies

$$\left|\frac{1+z\left(zf'(z)\right)''}{1+f''(z)}-\frac{z^2(1-\mu)f''(z)+zf'(z)}{\mu f(z)+z(1-\mu)f'(z)}\right|<\frac{1}{2}.$$

Then f is univalent in U.

3. Coefficient inequality

Theorem 3.1: If $f \in A$, then the function $f(x)$ belongs to the class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]|a_n| \le (1-\varepsilon). \tag{3.1}$$

The result is sharp for the functions

$$f(z)=z+\frac{(1-\varepsilon)}{n(n-2)+\mu(n-1)}z^n,\ n=2,3,\ldots$$

Proof: A function $f \in Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$ if and only if

$$\mathrm{Re}\left(\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}\right) > \varepsilon. \tag{3.2}$$

So by the definition 1.1

$$\left|\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}-1\right| \le 1-\varepsilon.$$

Now, let us show that this condition is satisfied under the hypothesis (3.1) of the theorem, we observe

$$\left|\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}-1\right| =$$

$$\left|\frac{\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]a_n z^n}{z+\sum_{n=2}^{\infty}\left[\mu(1-n)+n\right]a_n z^n}\right| \le \frac{\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]|a_n|}{1+\sum_{n=2}^{\infty}\left[\mu(1-n)+n\right]|a_n|}$$

The last expression is bounded above by $1-\varepsilon$ if

$$\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]|a_n| \le (1-\varepsilon)\left\{1+\sum_{n=2}^{\infty}\left[\mu(1-n)+n\right]|a_n|\right\},$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]|a_n| \le (1-\varepsilon).$$

Thus, the proof of Theorem 3.1 completed.

4. Growth and distortion theorem

Theorem 4.1: If $f \in A$, then the function $f(z)$ belongs to the class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$, then

$$r-\frac{1-\varepsilon}{\mu}r^2 \le \left|f\left(z\right)\right| \le r+\frac{1-\varepsilon}{\mu}r^2,$$

and

$$1-\frac{2\left(1-\varepsilon\right)}{\mu}r \le \left|f'\left(z\right)\right| \le 1+\frac{2\left(1-\varepsilon\right)}{\mu}r,$$

for $\left|z\right|=r<1$

Proof: In view of Theorem 3.1, radially yields

$$\sum_{n=2}^{\infty}\left|a_n\right| \le \frac{1-\varepsilon}{\mu} \tag{4.1}$$

Thus for $\left|z\right|=r<1$, and making use of (4.1) we have

$$\left|f\left(z\right)\right| \le r+r^2\sum_{n=2}^{\infty}\left|a_n\right| \le r+\frac{1-\varepsilon}{\mu}r^2,$$

and

$$\left|f\left(z\right)\right| \ge r-r^2\sum_{n=2}^{\infty}\left|a_n\right| \ge r-\frac{1-\varepsilon}{\mu}r^2.$$

Which proof the first part of the theorem. Also from the Theorem 3.1, it follows that

$$\frac{\mu}{2}\sum_{n=2}^{\infty}n\left|a_n\right| \le \sum_{n=2}^{\infty}\left[n\left(n-2\right)+\mu\left(n-1\right)\right]\left|a_n\right| \le 1-\varepsilon,$$

which gives

$$\sum_{n=2}^{\infty}n\left|a_n\right| \le \frac{2\left(1-\varepsilon\right)}{\mu}$$

Therefore, we have

$$\left|f'\left(z\right)\right| \le 1+r\sum_{n=2}^{\infty}n\left|a_n\right| \le 1+\frac{2\left(1-\varepsilon\right)}{\mu}r,$$

and

$$\left|f'\left(z\right)\right| \ge 1-r\sum_{n=2}^{\infty}n\left|a_n\right| \ge 1-\frac{2\left(1-\varepsilon\right)}{\mu}r$$

This completes the proof of Theorem 4.1.

5. Closer theorems

Let the functions $f_j\left(z\right), j=1,2,...I$, be defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n, \quad a_{n,j}\ge 0,\ z\in U \tag{5.1}$$

Closer theorems for the class $Q\left(\mu,\varepsilon\right),\ \varepsilon,\mu\in\left[0,1\right)$ are given by the following theorem

Theorem 5.1: Let the function $f_j\left(z\right)$ defined by (5.1) be in the class $Q\left(\mu,\varepsilon\right),\ \varepsilon,\mu\in\left[0,1\right)$ for every $j=1,2,...I$. Then the function $G\left(z\right)$ defined by

$$G(z) = z - \sum_{n=2}^{\infty} p_n z^n, \quad p_n \geq 0 \tag{5.2}$$

is a number of the class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$, where

$$p_n = \frac{1}{I}\sum_{j=1}^{I} a_{n,j} \qquad (n \geq 2).$$

Proof: Since $f_j(z) \in Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$, it follows from Theorem 3.1 that

$$\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left|a_{n,j}\right| \leq (1-\varepsilon),$$

for every $j = 1,2,\ldots I$. Hence

$$\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left|p_n\right| = \sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left\{\frac{1}{I}\sum_{j=1}^{I}\left|a_{n,j}\right|\right\}$$

$$\frac{1}{I}\sum_{j=1}^{I}\left(\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left|a_{n,j}\right|\right) \leq \frac{1}{I}\sum_{j=1}^{I}(1-\varepsilon),$$

which implies that $G(z) \in Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$.

Theorem 5.2: The class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$ is closed under convex combination

Proof: Suppose that the function $f_j(z)\ (j = 1,2)$ defined by (5.1) are in the class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$. It is suffices to prove that the function

$$H(z) = \psi f_1(z) + (1-\psi) f_2(z) \qquad (0 \leq \psi \leq 1) \tag{5.3}$$

is also in the class $Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$.

Since, for $0 \leq \psi \leq 1$,

$$H(z) = z + \sum_{n=2}^{\infty}\left\{\psi a_{n,1} + (1-\psi) a_{n,2}\right\} z^n,$$

we observe that

$$\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left|\psi a_{n,1} + (1-\psi) a_{n,2}\right| \leq \psi \sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left|a_{n,1}\right|$$

$$+(1-\psi)\sum_{n=2}^{\infty}\left[n(n-2)+\mu(n-1)\right]\left|a_{n,2}\right|$$

$$\leq \psi(1-\varepsilon) + (1-\psi)(1-\varepsilon) = (1-\varepsilon).$$

Hence $H(z) \in Q(\mu,\varepsilon),\ \varepsilon,\mu \in [0,1)$. This completes the proof of Theorem 5.1.

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