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The Green's Function Method Solutions to Solve of n^{th} -order Linear Differential Equations

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Abstract: In this research article, we present the Green's function approach of ordinary differential equations with initial and boundary conditions, we represented the differential equation by an integral equation. The text provides a sufficient theoretical basis to understand Green's function method, which is used to solve initial and boundary value problems involving linear ODEs and PDEs. The main result the construction of a Mathematica Package valid to calculate the explicit expression of the Green's function related to the two-point boundary value problem (2. 3), where the nth order linear operator Ln defined on (2. 1) has constant coefficients.

Keywords: Green's Functions, Ordinary linear Differential Equations.

طريقة حلول دالة جرين لحل المعادلات التفاضلية الخطية ذات من الرتبة النونية

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الملخص: في هذا المقال البحثي هذا، نقدم مقاربة دالة جرين للمعادلات التفاضلية العادية المتضمنة شروطاً ابتدائية وكذلك المتضمنة شروطاً حدية وتم تمثيل المعادلات التفاضلية بمعادلات تكاملية يوفر النص أساسًا نظريًا كافيًا لفهم طريقة دالة جرين، والتي تُستخدم لحل المشكلات ; القيمة الأولية والحدودية التي تتضمن المعادلات التفاضلية العادية والمعادلات التفاضلية الجزئية الخطية. النتيجة الرئيسية هي بناء حزمة ماثيمتكا صالحة لحساب التعبير الصريح لدالة جرين المتعلقة بمشكلة القيمة الحدية للنقطتين في المعادلة (2. 3)، حيث يكون للمؤثر الخطي من الدرجة النونية المحدد في المعادلة (2. 1) ذات المعاملات الثابتة.

الكلمات المفتاحية: دوال جرين، المعادلات التفاضلية الخطية العادية.

1. Introduction:

Consider a differential operator L of the following approach for initial and boundary value problems of ordinary differential equations.

$$Lu(t) = \left[A(t)\frac{d^2}{dt^2} + B(t)\frac{d}{dt} + C(t)\right] u(t), a < t < b \; (1.1)$$

Where $A(t)$ is continuously differentiable, positive function. It adjoin operator M is defined as (1)

$$Mv(t) = \frac{d^2}{dt^2}\,[A(t)\, v(t)] - \frac{d}{dt}\,[B(t)\, v(t)] + C(t)v(t), a < t < b \; (1.2)$$

Consider the integral

$$\int_a^b (v\, Lu - uMv)ds = \int_a^b \{\, v\,[\, A\, u'' + Bu' + Cu\,] - u\,[(Av)'' - (Bv)' + Cv]\,\}\, ds$$

$$= \int_a^b (A\, vu'' + Bv\, u')\, dt \; - \int_a^b u(A\, v)''dt \; + \int_a^b u(\, Bv)'\, dt \; (1.3)$$

Using integration by parts, we get

$$\int_a^b u\,(A\, v)'' = u(Av)'|_a^b - \int_a^b (A\, v)'\, u' dt = u(A\, v)'|_a^b - (Av)u'|_a^b - \int_a^b (A\, v)\, u''\, dt$$

$$\int_a^b u(\, Bv)'\, dt = u(Bv) - \int_a^b u'\,(Bv)\, dt$$

In such a case (1. 3) [2] becomes

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(1) C. Corduneanu. Integral Equations and Applications

(2) J. Kondo, Integral Equations

$$\int_a^b (v\,Lu - uMv)dt = \int_a^b \ (Au\,u'' + Bvu')dt -$$

$$\left(u(A\,v)'|_a^b - \left\{ A(v)\,u'|_a^b - \int_a^b \ (A\,v)\,u''\,dt \right\} \right) + u(Bv)|_a^b \ - \int_a^b \ u'\,(Bv)\,dt$$

$$= \ -u\,(Av' + A'v)|_a^b \ + \ (Av)u'|_a^b + u(Bv)|_a^b$$

$$= \ [\,A(vu' - uv') + uv(B - A')]_a^b$$

That is

$$\int_a^b (v\,Lu - uMv)dt = [\,A(vu' - uv') + uv(B - A')]_a^b \ (1.4)$$

Which is known as Green's formula for the operator.

2. Preliminaries:

In this section we study of the general two points nth –order differential equation[3]

$$L_n u(t) = \sigma(t), t \in \ J, U_i(u) = h_i, i = 1, \dots, n \ (2.1)$$

Where $U_i(u) \equiv \sum_{j=0}^{n-1} \left(\ \alpha_j^i u^{(j)} \ (a) + \ \beta_j^i \ u^{(j)} \ (b) \right), \ i = 1, \dots, n \ (2.2)$

And

$$L_n u(t) \equiv \ u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \cdots + a_{n-1}(t)u'(t) + a_n(t)\ u(t)t \in \ t \in \ J \ (2.3)$$

Being $\alpha_j^i, \ \beta_j^i$ and h_i real constants for all $i = 1, \dots, n \ and \ j = 0, \dots, n-1$ and

$$\sigma, a_k \in \mathcal{Y}^1(J, \mathbb{R}) \text{ for all } k = 1, \dots, n \ .$$

In this situation we look for solutions that belong to the space

(3) M.Rahman, Integral Equations and Their Applications

$$W^{n,1}(J) = \{ u \in \mathcal{G}^{n-1}(J,\mathbb{R}), u^{(n-1)} \in \mathcal{A}\,\mathcal{G} \} \ (2.4)$$

Definition (2-2): We say that g is a Green's function for problem (2. 1) – (2. 2) if it satisfies the following properties [4]:

$(g1)$ g is defined on the square $J \times J$ (except $t = s \ if \ n = 1$).

$(g2)$ For $k = 0, 1, \dots, n-1$, the partial derivatives $\frac{\partial^k g}{\partial t^k}$ exist and they are continuous on $J \times J$.

$(g3)$ $\frac{\partial^{n-1} g}{\partial t^{n-1}}$ and $\frac{\partial^n g}{\partial t^n}$. **Exist and are continuous on the triangles** $a \le s \le b \ and \ a \le t \le s \le b$.

$(g4)$ For each $s \in (a,b)$, the function $t \to g(t,s)$ is a solution of the differential equation

$$L_n y = 0 \ a.e.\ on\ [a,s) \bigcup (s,b]\ .That\ is,$$

$$\frac{\partial^n}{\partial t^n} g(t,s) + a_1(t)\frac{\partial^{n-1}}{\partial t^{n-1}} g(t,s) + \cdots + a_{n-1}\frac{\partial}{\partial t} g(t,s) + a_n(t) g(t,s) = 0$$

$$for\ all\ t \in J \setminus [s\} \ (2.5)$$

$(g5)$ For each $t \in (a,b)$ there exist the lateral limits

$$\frac{\partial^{n-1}}{\partial t^{n-1}} g(t^-,t) = \frac{\partial^{n-1}}{\partial t^{n-1}} g(t,t^+)\ and\ \frac{\partial^{n-1}}{\partial t^{n-1}} g(t,t^-) = \frac{\partial^{n-1}}{\partial t^{n-1}} g(t^+,t)\ (2.6)$$

and, moreover

$$\frac{\partial^{n-1}}{\partial t^{n-1}} g(t^+,t) - \frac{\partial^{n-1}}{\partial t^{n-1}} g(t,t^-) = \frac{\partial^{n-1}}{\partial t^{n-1}} g(t^-,t) - \frac{\partial^{n-1}}{\partial t^{n-1}} g(t,t^+) = 1\ (2.7)$$

$(g6)$ **For each** $s \in (a,b)$, **the function** $t \to g(t,s)$ **satisfies the boundary conditions** $U_i\big(g(.,s)\big) = 0, i = 1, \dots, n, i.e$

(4) **Alberto Cabada, Green's Functions in the Theory of Ordinary Differential Equations**

$$\sum_{j=0}^{n-1}\left(\alpha_j^i \frac{\partial^j}{\partial t^j} g(t,s) + \beta_j^i \frac{\partial^j}{\partial t^j} g(t,s)\right) = 0, i = 1, \ldots, n \ (2.8)$$

Example (2-3) [5]**: Consider, for any $m \in \mathbb{R}$, the second order initial value problem**

$$u''(t) + mu(t) = \sigma(t), u(0) = u'(0) = 0$$

To obtain the Green's function we only need to solve the following problem

$$r''(t) + mr(t) = 0, t \in \mathbb{R}, r(0) = 0, r'(0) = 1$$

It is immediate to verify that

$$r(t) = \begin{cases} \frac{sin(\sqrt{m}t)}{\sqrt{m}}, if\ m > 0 \\ t, if\ m = 0 \\ \frac{sin(\sqrt{-m}t)}{\sqrt{-m}}, if\ m < 0 \end{cases} \quad (2.9)$$

So, since $r(t) = g(t,0)$, the expression of the Green's function is deduced from expression

$$g(t,s) = g(t-s+a,a), if\ a \le s \le t \le b, and\ g(t,s) = 0\ otherwise$$

If we are interested in the periodic case,

$$u''(t) + mr(t) = 0, t \in \mathbb{R}, r(0) = r(1), r'(0) = r'(1) + 1\ (2.10)$$

The expression of the Green's function is obtaining by solving problem

$$L_n r(t) = 0, t \in J, r^{(i)}(a) = r^{(i)}(b),, i = 1, \ldots, n-2, r^{(n-1)}(a) = r^{(n-1)}(b) + 1$$

$$r''(t) + mr(t) = 0, t \in \mathbb{R}, r(0) = r(1),, r'(0) = r'(1) + 1\ (2.11)$$

Such equation has a unique solution given by

(5) V. D. S¸ eremet, Handbook of Green’s functions and matrices

$$r(t) = \begin{cases} \dfrac{\cos\left(\sqrt{m}(t-\frac{1}{2})\right)}{2\sqrt{m}\,\sin\frac{\sqrt{m}}{2}}, if\ m \neq 4k^2\pi^2, k = 0,1,\dots \\ -\dfrac{\cosh\left(\sqrt{-m}(t-\frac{1}{2})\right)}{2\sqrt{-m}\,\sinh\frac{\sqrt{-m}}{2}}, if\ m < 0 \end{cases} \quad (2.12)$$

Example (2-4) [6]: Consider the second order operator $L_n u(t) = u''(t) + t\,u'(t) + (\sin t)\,u(t)$,

Defined on the space $D(L) = \{u \in W^{2,2}([0,1],\mathbb{R}), u'(0) = u'(1) = 0\}$

In this case we have that

$$L^* v(t) = v''(t) - (t\,v(t))' + (\sin t)\,v(t) = v''(t) - tv'(t) + (-1 + sint)v(t) \quad (2.13)$$

The set of definition $D(L^*)$ of the adjoint operator consists of the functions $v \in W^{2,2}([0,1],\mathbb{R})$ that satisfy the following equality for all $u \in D(L)$:

$$(tv(t) - v'(t))\,u(t) + v(t)u'^{(t)}\Big|_{t=0} = (tv(t) - v'(t))\,u(t) + v(t)u'(t)\big|_{t=1} \quad (2.14)$$

Due to the fact that $u \in D(L)$ implies $u'^{(0)} = u'^{(1)}$, we conclude that the previous equality holds if and only if :

$$-v'(0)\,u(0) = \left(v(1) - v'(1)\right)u(1)\ for\ all\ u \in D(L) \quad (2.15)$$

That is

$$D(L^*) = \{v \in W^{2,2}([0,1],\mathbb{R}), v'(0) = v(1) - v'(1) = 0\}$$

Notice that if, instead of the Neumann boundary conditions, we study the Dirichlet case

$$D(L) = \{u \in W^{2,2}([0,1],\mathbb{R}), u(0) = u'(1) = 0\}$$

Then $D(L^*) = \{v \in W^{2,2}([0,1],\mathbb{R}), v'(0) = v(1) - v'(1) = 0\}$

For the periodic case

(6) V. D. S¸ eremet, Handbook of Green's functions and matrices.

$$D(L) = \{u \in W^{2,2}([0,1],\mathbb{R}), u(0) = u'(1);\ u'(0) = u'(1) = 0\} \ (2.16)$$

We conclude

$$D(L^*) = \{v \in W^{2,2}([0,1],\mathbb{R}), v'(0) = v(1) - v'(1)\}$$

Proposition (2-5)[7]: Assume $n \geq 2$ and that $a_k \in \mathbb{R}\ for\ all\ k \in \{1, \dots, n\}$. Suppose that the periodic boundary value problem:

$$L_n u(t) = \sigma(t), t \in J, u^{(i)}(a) = u^{(i)}(b), i = 1, \dots, n-1 \ (2.17)$$

Has a unique solution for all $\sigma \in \mathcal{G}^I(J,\mathbb{R})$. Then the following property is fulfilled:

$$\frac{\partial g}{\partial t}(t,s) = -\frac{\partial g}{\partial t}(t,s), for\ all\ t,s \in J \ (2.18)$$

Proof: Suppose that function σ is differentiable. Let u be a solution of the considered periodic problem $u \in \mathcal{G}^n(J,\mathbb{R}) and\ v \equiv u'$ is a solution of

$$L_n v(t) = \sigma'(t), t \in J$$

$$v^{(i)}(a) - v^{(i)}(b) = 0, i = 0, \dots, n-2$$

$$v^{(n-1)}(a) - v^{(n-1)}(b) = \sigma(a) - \sigma(b)$$

Therefore, from the properties of the Green's function of the periodic problem shown in this section, we deduce that

$$v(t) = \int_a^b g(t,s)\sigma'(s)\,ds + g(t,a)\big(\sigma(a) - \sigma(b)\big) \ (2.19)$$

So, by integration by parts and form the fact that $g(t,a) = g(t,b)$, we have that

$$v(t) = g(t,b)\sigma(b) - g(t,a)\sigma(a) - \int_a^t \frac{\partial g}{\partial s}(t,s)\sigma(s)\,ds - \int_t^b \frac{\partial g}{\partial s}(t,s)\sigma(s)\,ds$$

$$+g(t,a)\big(\sigma(a) - \sigma(b)\big) = -\int_a^b \frac{\partial g}{\partial s}(t,s)\,\sigma(s)\,ds \ (2.20)$$

On the other hand, using that $n \geq 2$, we deduce

(7) M. Renardy, R. C. Rogers, An introduction to partial differential equations

$$v(t) = u'(t) = \frac{\partial}{\partial t} \int_a^t g(t,s)\sigma(s)\, ds + \frac{\partial}{\partial t}\int_t^b g(t,s)\, \sigma(s)\, ds = \int_a^b \frac{\partial g}{\partial t}(t,s)\, \sigma(s)\, ds \ (2.21)$$

Since the differentiable functions are dense in $\mathcal{G}^2(J, \mathbb{R})$, we conclude that

$$\frac{\partial G}{\partial t}(t,s) = -\frac{\partial G}{\partial t}(t,s) \ \#$$

Lemma (2-6)[(8)]**:** Suppose that the general n^{th} –order linear differential operator Ln defined in (2. 3) has constant coefficients. Then problem (2. 1) has a unique solution for every $\sigma \in (\mathcal{G}^I[a,b], \mathbb{R})$ and $h_i \in \mathbb{R}, i = 1, \dots, n$ if and only if problem

$$\tilde{L}_n u(t) = \tilde{\sigma}(t), t \in [c,d], \tilde{U}_i(u) = h_i, i = 1, \dots, n$$

Has a unique solution for every $\tilde{\sigma} \in (\mathcal{G}^I[a,b], \mathbb{R})$.

Moreover, denoting as the corresponding related g *and* $\tilde{g}$ Green's functions, we have that the following equality fulfilled

$$\tilde{g}(t,s) = \left(\frac{d-s}{b-a}\right)^{n-1} g\left(\frac{b-a}{d-c}(t-c) + a, \frac{b-a}{d-c}(s-c) + a\right),$$

$$for\ all\ (t,s) \in [c,d] \times [c,d] \ (2.22)$$

Proof: First note that $\tilde{\sigma} \in (\mathcal{G}^I[a,b], \mathbb{R})$ if and only if there is $\sigma \in (\mathcal{G}^I[a,b], \mathbb{R})$

Such that $\tilde{\sigma}(t) = \sigma\left(\frac{b-a}{d-c}(t-c) + a\right)\ for\ all\ t \in [c,d]$

The first part of the proof follows form this fact and the direct verification that u is a solution of (2. 3) if and only if

$$v(t) = \left(\frac{d-c}{b-a}\right)^{n} u\left(\frac{b-a}{d-c}(t-c) + a\right), t \in [c,d] \ (2.23)$$

Satisfies that

$$\tilde{L}_n v(t) = \tilde{\sigma}(t), t \in [c,d], \tilde{U}_i(v) = h_i, i = 1, \dots, n$$

The second part of the proof is given by the following equalities for all $t \in [c,d]$

(8) M. Rahman, Integral Equations and Their Applications

$$v(t) = \left(\frac{d-c}{b-a}\right)^n u\left(\frac{b-a}{d-c}(t-c)+a\right)$$

$$= \left(\frac{d-c}{b-a}\right)^n \int_a^b g\left(\frac{b-a}{d-c}(t-c)+a, s\right)\sigma(s)\, ds$$

$$= \left(\frac{d-c}{b-a}\right)^{n-1} \int_c^d g\left(\frac{b-a}{d-c}(t-c)+a, \frac{b-a}{d-c}(\tau-c)\right)\sigma\left(\frac{b-a}{d-c}(\tau-c)+a\right)d\tau \quad (2.24)$$

3. Main Result:

Sometimes the difficulty of the calculations to be made in the study of the Green's function depends strongly on the extremes of the interval. In general it is easier to obtain the explicit expression of the considered problem in, for instance, the intervals $[0,1]$ or $[0,2\pi]$ than in the general one $[a,b]$. In the next result we show that if the linear operator L_n has constant coefficients, by means of a simple change of variable, we can chose the interval where we can deal with. The arguments extend to the general situation the case studied in [5, Lemma 2. 4]. To this end we define:

$$\breve{L}_n u(t) \equiv u^{(n)}(t) + \widetilde{a_1}\, u^{(n-1)}(t) + \cdots + \tilde{a}_{n-1}\, u'^{(t)} + \tilde{a}_n\, u'^{(t)}, t \in [c,d],$$

Here $\widetilde{a}_j = a_j \left(\frac{b-a}{d-c}\right)^j, j = 1,..,n$

and $\breve{U}_i(u) \equiv \sum_{j=0}^{n-1}(\breve{\alpha}_j^i\, u^{(j)}(c) + \breve{\beta}_j^i\, u^{(j)}(d)), i = 1,..,n$

With $\breve{\alpha}_j^i = \alpha_j^i \left(\frac{b-a}{d-c}\right)^{n-j}, \breve{\beta}_j^i = \beta_j^i \left(\frac{b-a}{d-c}\right)^{n-j}, j = 1,..,n-1, i = 1,..,n$

Lemma (3-1) [(9)]**: Suppose that the general n^th-order linear differential operator L_n defined in (2.3) has constant coefficients. Then problem (2.1) has a unique solution for every $\sigma \in \mathfrak{J}^1([a,b],\mathbb{R})$ and $h_i \in \mathbb{R}, i = 1, \dots, n$ if and only if problem**

$$\tilde{L}_n u(t) = \tilde{\sigma}(t), t \in [a,d], \tilde{U}_i(u) = h_i, i = 1, \dots, n$$

Has a unique solution for every $\tilde{\sigma} \in \mathfrak{J}^1([c,d],\mathbb{R})$.

Moreover, denoting as g and $\tilde{g}$ the corresponding related Green's functions, we have that the following equality is fulfilled

(9) P. Hartman, Ordinary differential equations

$$\tilde{g}(t,s) = \left(\frac{b-a}{d-a}\right)^{n-1} g\left(\frac{b-a}{d-c}(t-c)+a, \frac{b-a}{d-a}(s-c)+a\right)$$

For all $(t,s) \in [c,d] \times [c,d]$.

Proof: First note that $\tilde{\sigma} \in \mathfrak{J}^1([c,d],\mathbb{R})$, if and only if there is $\sigma \in \mathfrak{J}^1([a,b],\mathbb{R})$ such that $\tilde{\sigma}(t) = \sigma\left(\frac{b-a}{d-c}(t-c)+a\right)$ For all $t \in [c,d]$

The first part of the proof follows from this fact and the direct verification that u

Is a solution of (2.1) if and only if

$$v(t) = \left(\frac{d-c}{b-a}\right)^{n} u\left(\frac{b-a}{d-c}(t-c)+a\right), t \in [c,d]$$

Satisfies that $\tilde{L}_n v(t) = \tilde{\sigma}(t), t \in [c,d], \tilde{U}_i(v) = h_i, i = 1, \dots, n$

The second part of the proof is given by the following equalities for all $t \in [c,d]$

Lower and Upper Solutions(10)

$$v(t) = \left(\frac{d-c}{b-a}\right)^{n} u\left(\frac{b-a}{d-c}(t-c)+a\right)$$

$$= \left(\frac{d-c}{b-a}\right)^{n} \int_a^b g\left(\frac{b-a}{d-c}(t-c)+a, s\right) \sigma(s) ds$$

$$= \left(\frac{d-c}{b-a}\right)^{n-1} \int_a^b g\left(\frac{b-a}{d-c}(t-c)+a, \frac{b-a}{d-c}(\tau-c)+a\right) \sigma\left(\frac{b-a}{d-c}(\tau-c)+a\right) d\tau$$

Conclusion:

By using the Green's function approach of ordinary differential equations with initial and boundary conditions, and consider the nth order differential operator

$$\mathcal{L} = \frac{d^n}{d\,t^n} + a_{n-1}\frac{d^{n-1}}{d\,t^{n-1}} + a_{n-2}\frac{d^{n-2}}{d\,t^{n-2}} + \cdots + a_1\frac{d}{d\,t} + a_0$$

We want to find the general solution to the differential equation $\mathcal{L}[x(t)] = f(t)$. Where the forcing function $f(t)$ turns on at $t = 0$ i,e $f(t) = 0$ for $t \le 0$. The general solution will be the sum $x(t) = x_h(t) + x_p(t)$ where (i) $x_h(t)$ is a homogeneous solution, satisfying

(10) Y. A. Melnikov, M. Y. Melnikov, Green's functions.

$\mathcal{L}\,[x_h(t)] = 0$, with arbitrary initial $x_h(t)$ and its first $n-1$ derivatives at $t = 0$ and (ii) $x_p(t)$ is a particular solution, satisfying

$\mathcal{L}\,[x_h(t)] = f(t)$, With the initial conditions that $x_p(t)$ and its first $n-1$

derivatives at $t = 0$.

We will assume that we know how to solve for the general homogeneous solution $x_h(t)$.

References:

[1] C. Corduneanu. Integral Equations and Applications. Cambridege, England:
Cambridge University Press (1991).

[2] J. Kondo, Integral Equations. Oxford, England: Clarendon Press (1992).

[3] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Integral Equations and Inverse Theory. Ch. 18 in Numerical Recipes in Fortran: The Art of Scientific Computing, 2nd ed. Camabride, England: Cambridge University Press, PP. 779-817 (1992).

[4] Alberto Cabada, Green's Functions in the Theory of Ordinary Differential Equations, Book. November 2014.

[5] M. Rahman, Integral Equations and Their Applications, WIT, Southampton, Boston (2007).

[6] V. D. S¸ eremet, Handbook of Green’s functions and matrices. With 1 CD-ROM (Windows and Macintosh). WIT Press, Southampton, 2003.

[7] M. Renardy, R. C. Rogers, An introduction to partial differential equations. Texts in Applied Mathematics 13 (Second edition ed.). New York, Springer-Verlag. (2004).

[8] M. Rahman, Integral Equations and Their Applications, WIT, Southampton, Boston (2007).

[9] Y. A. Melnikov, M. Y. Melnikov, Green’s functions. Construction and applications. de Gruyter Studies in Mathematics, 42. Walter de Gruyter & Co., Berlin, 2012.

[10] P. Hartman, Ordinary differential equations. John Wiley & Sons, Inc., New York-London-
Sydney, 1964.

[11] J. Mawhin, Twenty years of ordinary differential equations through twelve Oberwolfach meetings. Results Math. 21 (1992), 1-2, 165–189.