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(STRONGLY) $\pi -$REGULAR RINGS RELATIVE TO RIGHT IDEAL

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Abstract: In this paper we study the notion of $\pi -$regular and strongly $\pi -$regular rings relative to right ideal. We provide several characterizations of this rings and study their properties. It is shown that every ring R is $\pi -$regular relative to any maximal right ideal of R. Also, we find necessary and sufficient conditions to be a ring R satisfies the d.c.c. on chains of the form $Ra \supseteq Ra^2 \supseteq \Lambda$ relative to ideal for every $a \in R$. New results obtained include necessary and sufficient conditions for a ring to be $\pi -$regular, strongly $\pi -$regular and $P -$potent relative to right ideal.

Keywords: Regular and $\pi -$ regular rings, Strong $\pi -$ regular rings, Potent and Semi- potent rings, Annihilator, Ring satisfies the d.c.c. on chains.

الحلقات $\pi-$المنتظمة و $\pi-$قوية الانتظام بالنسبة لمثالي يميني

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الملخص: في هذا البحث درسنا مفهوم الحلقات $\pi-$المنتظمة والحلقات $\pi-$قوية الانتظام بالنسبة لمثالي يميني، حيث أوردنا عدداً من التوصيفات لهذه الحلقات ودرسنا أيضاً العديد من خصائصها. أيضاً أثبتنا أن كل حلقة R تكون $\pi-$منتظمة بالنسبة لأي مثالي يميني أعظمي للحلقة R. وأيضاً أوجدنا الشروط اللازمة والكافية كي تحقق الحلقة R شرط انقطاع السلاسل المتناقصة من الشكل $Ra \supseteq Ra^2 \supseteq \Lambda$ بالنسبة لمثالي وذلك لأجل كل $a \in R$ وقد حصلنا على نتائج جديدة من ضمنها الشروط اللازمة والكافية لأجل الحلقة كي تكون $\pi-$منتظمة، $\pi-$قوية الانتظام و $P-$جامدة بالنسبة لمثالي يميني.

الكلمات المفتاحية: الحلقات المنتظمة و $\pi-$المنتظمة، الحلقات $\pi-$قوية الانتظام، الحلقات الجامدة وشبه الجامدة، العادم، انقطاع السلاسل المتناقصة.

1- Introduction.

The notion of rings relative to right ideal, was first introduced by V. A. Andrunakievich and Yu. M. Ryabukhin in 1987 [2]. They studied the concepts of quasi- regularity and primitivity of rings relative to right ideal. Later, V. A. Andrunakievich and A. V. Andrunakievich in 1991 [1], studied the concept of regularity of rings relative to right ideal as generalization of (Von Neumann) regular ring (also known as $P-$regular rings). A number of interesting papers have been published on this concept in recent years, e.g., [1], [2], [3], [5]. A ring R is called regular relative to right ideal $P \neq R$, if for every $a \in R$ there exists $b \in R$ such that $aba - a \in P$ and $abP \subseteq P$ [2]. In [2] it is proved that any ring R is regular relative to any maximal right ideal of R, [2, Corollary 2]. Also, it is proved that a ring R is regular relative to right ideal $P \neq R$ if and only if for every $a \in R$, $aR + P = eR + P$ where $e \in R$ is idempotent relative to P. In 2011, P. Dheena and S. Manivasan [3] studied quasi- ideals of a $P-$regular near- rings. In [5], H. Hakmi continue study $P-$regular and $P-$potent rings. In this paper, we study the notion of $\pi-$regular and strongly $\pi-$regular rings relative to right ideal. In section 2, we provide some characterizations of $\pi-$regular rings relative to right ideal and investigate its properties. We prove that a ring R is $\pi-$regular relative to right ideal $P \neq R$ if and only if for every $a \in R$ there is a positive integer n such that $a^n R + P = eR + P$ for some $P-$idempotent $e \in R$. We show that every ring R is $\pi-$regular relative to any maximal right ideal of R. Also, we prove that the Jacobson radical of $\pi-$regular ring relative to right ideal $P \neq R$ is $P-$nil. In section 3, we study the rings R that satisfy the d.c.c. on chains of the form

$$aR + P \supseteq a^2R + P \supseteq a^3R + P \supseteq \Lambda \supseteq \Lambda$$

$$aR + RP \supseteq a^2R + RP \supseteq a^3R + RP \supseteq \Lambda \supseteq \Lambda$$

where $P \neq R$ is a right ideal of R. We prove that the chain

$$aR + RP \supseteq a^2R + RP \supseteq a^3R + RP \supseteq \Lambda \supseteq \Lambda$$

is stationary if and only if the chain

$$Ra + RP \supseteq Ra^2 + RP \supseteq Ra^3 + RP \supseteq \Lambda \supseteq \Lambda \text{ is}$$

stationary. Also, we provide some characterizations of strongly $\pi-$regular rings relative to ideal and investigate its properties. In section 4, we study the rings $P-$potent rings relative to right ideal. In this section, we provide some characterizations of $P-$potent rings relative to right ideal and investigate its properties.

Throughout in this paper rings R, are associative with identity unless otherwise indicated. We denote the Jacobson radical of a ring R by $J(R)$. We also denote $r(a) = \{x : x \in R; ax = 0\}$ the right annihilator for every $a \in R$.

2. $\pi-$ Regular Rings Relative to Right Ideal.

Recall that a ring R is $\pi-$regular [7], if for every $a\in R$ there exists $b\in R$ such that $a^n=a^nba^n$ for some positive integer n. Let R be a ring and $P\neq R$ be a right ideal of R, we say that a ring R is $\pi-$regular relative to right ideal P if for every $a\in R$ there exists $b\in R$ such that $a^n-a^nba^n\in P$ and $a^nbP\subseteq P$. Recall that an element $e\in R$ is $P-$idempotent [1], if $e^2-e\in P$ and $eP\subseteq P$. Note that for $P=0$ the concept, $\pi-$regular ring relative to right ideal P and the concept $\pi-$regular ring are corresponding.

Theorem 2.1. Let R be a ring and $P\neq R$ be a right ideal of R. Then the following statements are equivalent:

1. A ring R is $\pi-$regular relative to right ideal P.
2. For every $a\in R$ there exists $b\in R$ and $P-$idempotent $e\in R$ such that
$$a^nR+P=eR+P$$
for some positive integer n.
3. For every $a\in R$ there exists $b\in R$ such that $(a^nR+P)\cap((1-a^nb)R+P)=P$ for some positive integer n.

Proof. $(1)\Rightarrow(2)$. Let $a\in R$, then $a^n-a^nba^n\in P$ and $a^nbP\subseteq P$ for some $b\in R$ and positive integer n. Let $e=a^nb$, then $e\in R$ is $P-$idempotent and
$$eR+P=a^nbR+P\subseteq a^nR+P$$
On the other hand, since $a^n-a^nba^n\in P$, $a^n=a^nba^n+p_0$ where $p_0\in P$ and
$$a^nR+P\subseteq a^nba^nR+p_0P+P\subseteq a^nbR+PR+P\subseteq eR+P$$
$(2)\Rightarrow(1)$. Let $a\in R$, then by assumption there exists $P-$idempotent $e\in R$ such that $a^nR+P=eR+P$ for some positive integer n. So $a^n=ex+p_1$ for some $x\in R$, $p_1\in P$ and $e=a^nb+p_2$ for some $b\in R$, $p_2\in P$. Since $e\in R$ is $P-$idempotent, $e^2-e\in P$ and $eP\subseteq P$. Suppose that $e^2=e+p_3$ where $p_3\in P$, then

$$ea^n=e^2x+ep_1=(e+p_3x)+ep_1$$
$$(a^nb+p_2)a^n=ex+p_3x+ep_1$$
$$a^nba^n=a^n-p_1+p_3x+ep_1-p_2a^n$$
$$a^nba^n-a^n=(-p_1+ep_1)+(p_3x-p_2a^n)\in P+PR\subseteq P$$

On the other hand, since $a^nb=e-p_2$, for every $t\in P$
$$a^nbt=(e-p_2)t=et-p_2t\in P+PR\subseteq P$$
so $a^nbP\subseteq P$.

$(1) \Rightarrow (2)$. Let $a \in R$, then $a^n - a^n b a^n \in P$ and $a^n bP \subseteq P$ for some $b \in R$ and positive integer n. Since $a^n bP \subseteq P$, by [5, Lemma 2.1]

$$(a^n R + P) \cap ((1 - a^n b)R + P) = (a^n - a^n b a^n)R + P = P$$

$(2) \Rightarrow (1)$. Let $a \in R$, then by assumption there is $b \in R$ such that

$$(a^n R + P) \cap ((1 - a^n b)R + P) = P$$

for some positive integer n. Since

$$(a^n - a^n b a^n)R = a^n (1 - b a^n)R \subseteq a^n R \subseteq a^n R + P$$

$$(a^n - a^n b a^n)R = (1 - a^n b) a^n R \subseteq (1 - a^n b)R \subseteq (1 - a^n b)R + P$$

implies that $(a^n - a^n b a^n)R \subseteq P$, so $a^n - a^n b a^n \in P$. Suppose that $a^n = a^n b a^n + p_0$ where $p_0 \in P$, then $a^n b = a^n b a^n b + p_0 b$, so for every $t \in P$

$$a^n bt = a^n b a^n bt + p_0 bt \in a^n R + P$$

On the other hand, since $1 = a^n b + (1 - a^n b)$,

$t = a^n bt + (1 - a^n b)t$, so $a^n bt = t - (1 - a^n b)t \in (1 - a^n b)R + P$

by assumption implies that

$$a^n bt \in (a^n R + P) \cap ((1 - a^n b)R + P) = P$$

so $a^n bP \subseteq P$. Thus our proof is completed.

Note that in Theorem 2.1 and for $P = 0$ we derive the following:

Corollary 2.2. For any ring R the following statements are equivalent:

1- A ring R is $\pi-$regular.

2- For every $a \in R$ there exists idempotent $e \in R$ such that $a^n R = eR$ for some positive integer n.

3- For every $a \in R$ there exists $b \in R$ such that $a^n R \cap (1 - a^n b)R = 0$ for some positive integer n.

Lemma 2.3. Any ring R is $\pi-$regular relative to every maximal right ideal of R.

Proof. Let M be a maximal right ideal of R and $a \in R$. If $a \in M$, then for every positive integer n, $a^n = a a^{n-1} \in MR \subseteq M$ so for every $b \in R$,

$$a^n - a^n b a^n \in M + MR \subseteq M$$

and $a^n bM \subseteq MR \subseteq M$. Let $a \notin M$, assume that n is a smallest positive integer with $a^n \notin M$, then $R = a^n R + M$ and so $1 = a^n x + m$ for some $x \in R$, $m \in M$ and for every $y \in R$, $y = a^n xy + my$ so $a^n xy - y = -my \in MR \subseteq M$. Then for $y = a^n$, $a^n - a^n x a^n \in M$ and for every $t \in M$, $a^n xt = (1 - m)t = t - mt \in M + MR \subseteq M$. This shows that $a^n xM \subseteq M$. Thus our proof is completed.

Theorem 2.4. Let R be a ring, $P \neq R$ be a right ideal of R and Q be a maximal right ideal of R. If R is $\pi-$regular relative to right ideal P, then R is $\pi-$regular relative to right ideal $Q \cap P$.

Proof. Suppose that R is $\pi-$regular relative to right ideal P. Let $a \in R$, then there exists $b \in R$ and positive integer n such that $a^n - a^n b a^n \in P$, $a^n bP \subseteq P$. Now we have two states:

1- $a^n - a^n b a^n \notin Q$.

2- $a^n - a^n b a^n \in Q$.

State (1). Suppose that $a^n - a^n b a^n \notin Q$, since Q is maximal, $R = (a^n - a^n b a^n)R + Q$ so $1 = (a^n - a^n b a^n)x + q$ for some $q \in Q$ and for every $u \in R$

$$(a^n b a^n - a^n)xu - u = -qu \in QR \subseteq Q \qquad (1)$$

so for $u = a^n b a^n - a^n$ implies that

$$(a^n b a^n - a^n)x(a^n b a^n - a^n) - (a^n b a^n - a^n) \in Q$$

Since $a^n b a^n - a^n \in P$ follows

$$(a^n b a^n - a^n)x(a^n b a^n - a^n) - (a^n b a^n - a^n) \in P \cap Q$$

$$(a^n b a^n - a^n)x(a^n - a^n b a^n) + (a^n b a^n - a^n) \in P \cap Q$$

$$(a^n b a^n x - a^n x)(a^n - a^n b a^n) + (a^n b a^n - a^n) \in P \cap Q$$

$$a^n (b a^n x - b a^n x a^n b - x + x a^n b + b)a^n - a^n \in P \cap Q$$

Let $z = b a^n x - b a^n x a^n b - x + x a^n b + b$, then $a^n z a^n - a^n \in P \cap Q$. Now we prove that $a^n z(P \cap Q) \subseteq P \cap Q$. For every $t \in P \cap Q$ we have

$$a^n zt = a^n (b a^n x - b a^n x a^n b - x + x a^n b + b)t =$$

$$= (a^n b a^n - a^n)xt - (a^n b a^n x a^n - a^n x a^n)bt + a^n bt =$$

$$= (a^n b a^n - a^n)x(t - a^n bt) - (t - a^n bt) + t$$

Since $a^n b a^n - a^n \in P$, $(a^n b a^n - a^n)x(t a^n bt) \in PR \subseteq P$ and

$$a^n bt \in a^n b(Q \cap P) \subseteq a^n bP \subseteq P$$

which implies that $a^n zt \subseteq P$. In addition to, by (1) and for $u = t - a^n bt \in R$ implies that

$$a^n zt = (a^n b a^n - a^n)x(t - a^n bt) - (t - a^n bt) + t \in Q + Q \subseteq Q$$

So $a^n zt \in P \cap Q$, thus $a^n z(P \cap Q) \in P \cap Q$. This shows that R is $\pi-$regular relative to right ideal $P \cap Q$.

State (2). Suppose that $a^n - a^n b a^n \in Q$, since $a^n - a^n b a^n \in P$, $a^n - a^n b a^n \in P \cap Q$. If $a^n bP \subseteq Q$, then $a^n bP \subseteq P \cap Q$ and so

$$a^n b(P \cap Q) \subseteq a^n bP \subseteq P \cap Q$$

This shows that R is $\pi-$regular relative to right ideal $P \cap Q$.

Suppose that $a^n bP \not\subset Q$, then $a^n bPa^n \not\subset Q$, hence if $a^n bPa^n \subseteq Q$ implies that

$$a^n bPa^n bP = (a^n bPa^n)bP \subseteq QR \subseteq Q$$

So $(a^n bP)^2 \subseteq Q$. Since the right ideal Q is maximal, Q is prime which implies that $a^n bP \subseteq Q$ a contradiction. Thus, there exists $p_0 \in P$ such that $a^n bp_0 a^n \not\subset Q$, since $a^n ba^n - a^n \in P \cap Q$ and $a^n bP \subseteq P$ follows that

$$a^n ba^n - a^n + a^n bp_0 a^n \in P + PR \subseteq P$$

$$a^n bP + a^n bp_0 P \in P + PR \subseteq P$$

$$a^n ba^n - a^n + a^n bp_0 a^n \notin Q$$

This shows that $a^n(b + bp_0)a^n - a^n \in P$, $a^n(b + bp_0)P \subseteq P$ and

$$a^n(b + bp_0)a^n - a^n \notin Q$$

Let $z_0 = b + bp_0 \in R$, then we have $a^n z_0 a^n - a^n \in P$, $a^n z_0 P \subseteq P$ and

$$a^n z_0 a^n - a^n \notin Q$$

by state (1) implies that R is $\pi-$regular relative to right ideal $P \cap Q$.

Theorem 2.5. Every ring R is $\pi-$regular relative to the intersection of finite number of maximal right ideals of R.

Proof. Follows immediately from Lemma 2.3 and Theorem 2.4.

Let R be a ring and $P \neq R$ be a right ideal of R. We say that an element $a \in R$ is $P-$nilpotent if $a^n \in P$ for some positive integer n. Also, we say that a subset A of a ring R is $P-$nil if every element of A is $P-$nilpotent.

Lemma 2.6. Let R be a ring and $P \neq R$ be a right ideal of R. If R is $\pi-$regular relative to right ideal P, then $J(R)$ is $P-$nil.

Proof. Let $a \in J(R)$, then $a^n - a^n ba^n \in P$ and $a^n bP \subseteq P$ for some $b \in R$ and positive integer n. Then $a^n - a^n ba^n = p_0$ where $p_0 \in P$. So $a^n(1 - ba^n) = p_0$, since $a \in J(R)$, $1 - ba^n$ is invertible, i.e., $a^n = p_0 x \in PR \subseteq P$ for some $x \in R$.

Lemma 2.7. [5, Lemma 2.3]. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements hold:

1- For every $a \in R$ the set $r_P(a) = \{x : x \in R; ax \in P\}$ is a right ideal of R.
2- $r_P(1) = P$.
3- For every $a \in R$, $aP \subseteq P$ if and only if $P \subseteq r_P(a)$.
4- For every $a \in R$, $r_P(a) = R$ if and only if $a \in P$.

Theorem 2.8. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:

1- A ring R is $\pi-$regular relative to right ideal P.

2- For every $a \in R$ there exists $b \in R$ such that $a^n R + P = r_P(1-a^n b)$ for some positive integer n.

Proof. $(1) \Rightarrow (2)$. Let $a \in R$, then $a^n - a^n b a^n \in P$ and $a^n bP \subseteq P$ for some $b \in R$ and positive integer n. Since $(1-a^n b)a^n = a^n - a^n b a^n \in P$, $a^n \in r_P(1-a^n b)$, this shows that $a^n R \subseteq r_P(1-a^n b)$. On the other hand, since for every $t \in P$,

$$(1-a^n b)t = t - a^n bt \in P + a^n bP \subseteq P$$

$(1-a^n b)P \subseteq P$, so $a^n R + P \subseteq r_P(1-a^n b)$.

Let $x \in r_P(1-a^n b)$, then $(1-a^n b)x \in P$, so $x = a^n bx + p_0 \in a^n R + P$ where $p_0 \in P$. Thus $r_P(1-a^n b) \subseteq a^n R + P$.

$(2) \Rightarrow (1)$. Let $a \in R$, then by assumption there exists $b \in R$ such that

$$a^n R + P = r_P(1-a^n b)$$

for some positive integer n. Since $a^n \in r_P(1-a^n b)$, $a^n - a^n b a^n \in P$.

Let $t \in P \subseteq r_P(1-a^n b)$, then $(1-a^n b)t = p_1$ where $p_1 \in P$. So $a^n bt = t - p_1 \in P$ and so $a^n bP \subseteq P$. Thus our proof is completed.

Note that in Theorem 2.8 and for $P = 0$ we derive the following:

Corollary 2.9. For any ring R the following statements are equivalent:

1 – A ring R is $\pi-$regular.

2 – For every $a \in R$ there exists $b \in R$ such that $a^n R = r(1-a^n b)$ for some positive integer n.

3 . Strongly $\pi-$Regular Rings.

Let R be a ring and $P \neq R$ be a right ideal of R. For every $a \in R$ we define the following chains of the form:

(1) Descending chain of principal right ideals:

$$aR \supseteq a^2 R \supseteq a^3 R \supseteq \Lambda \qquad (3.1)$$

(2) Descending chain of principal left ideals:

$$Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \Lambda \qquad (3.2)$$

(3) Descending chain of right ideals:

$$aR + P \supseteq a^2 R + P \supseteq a^3 R + P \supseteq \Lambda \qquad (3.3)$$

(4) Descending chain of right ideals:

$$aR + RP \supseteq a^2 R + RP \supseteq a^3 R + RP \supseteq \Lambda \qquad (3.4)$$

(5) Descending chain of left ideals:

$$Ra + RP \supseteq Ra^2 + RP \supseteq Ra^3 + RP \supseteq \Lambda \qquad (3.5)$$

(6) Ascending chain of right ideals:

$$r(a) \subseteq r(a^2) \subseteq r(a^3) \subseteq \Lambda \qquad (3.6)$$

(7) Ascending chain of left ideals:

$$\lambda(a) \subseteq \lambda(a^2) \subseteq \lambda(a^3) \subseteq \Lambda \qquad (3.7)$$

It is known that the chain (3.1) is stationary for every $a \in R$ if and only if the chain (3.2) is stationary for every $a \in R$, [4, Theorem 3.16]. A ring R satisfy the d.c.c. on chain (3.1) for every $a \in R$ are often called a strongly $\pi-$regular ring. It is clear that if for some $a \in R$ the chain (3.1) is stationary, then the chain (3.7) is stationary. Also, if for some $a \in R$ the chain (3.2) is stationary, then the chain (3.6) is stationary. It is easy to see that, if for some $a \in R$ the chain (3.1) is stationary, then the chains (3.3) and (3.4) are stationary. Also, if for some $a \in R$ the chain (3.2) is stationary, then the chain (3.5) is stationary.

Lemma 3.1. Let R be a ring and $a \in R$. Then the following are equivalent:

1- The chain (3.1) is stationary.

2- $R = aR + r(a^n)$ for some positive integer n.

3- $R = a^n R + r(a^n)$ for some positive integer n.

Proof. $(1) \Rightarrow (2)$. Assume that the chain (3.1) is stationary. Then $a^n R = a^{n+1} R$ for some positive integer n. So $a^n = a^{n+1}x$ for some $x \in R$ and $a^n(1-ax) = 0$, this shows that $1-ax \in r(a^n)$ and so $R = a^n R + r(a^n)$.

$(2) \Rightarrow (1)$. Suppose that $R = aR + r(a^n)$ for some positive integer n. Then $1 = ax + y$ for some $x \in R$ and $y \in r(a^n)$, so

$$a^n = a^{n+1}x + a^n y = a^{n+1}x \in a^{n+1}R$$

this shows that $a^n R = a^{n+1} R$. Thus, the chain (3.1) is stationary.

$(1) \Rightarrow (2)$ .Assume that the chain (3.1) is stationary. Then $a^n R = a^{n+1} R = a^{2n} R$ for some positive integer n. So $a^n = a^{2n}x$ for some $x \in R$ and $a^n(1-a^n x) = 0$, this shows that $1-a^n x \in r(a^n)$ and so $R = a^n R + r(a^n)$. $(3) \Rightarrow (2)$. Is obvious.

Theorem 3.2. Let R be a ring. Then the following statements are equivalent:

1 – The ring R is strongly $\pi-$regular.

2 – For every $a \in R$ there exists a positive integer n such that $R = aR + r(a^n)$.

$2'$ – For every $a \in R$ there exists a positive integer m such that $R = Ra + \lambda(a^m)$.

3 – For every $a \in R$ there exists a positive integer n such that $R = a^n R \oplus r(a^n)$.

$3'$ – For every $a \in R$ there exists a positive integer m such that $R = Ra^m \oplus \lambda(a^m)$.

Proof. $(1) \Leftrightarrow (2)$ .By Lemma 3.1. $(1) \Rightarrow (3)$. Let $a \in R$, then the chain (3.1) is stationary, so $a^n R = a^{n+1} R = a^{2n} R$ for some positive integer n, by Lemma 3.1 $R = a^n R + r(a^n)$. Assume that $a^n R \cap r(a^n) \neq 0$, then there exists $x \in R$, $x \neq 0$ such that $a^n x = 0$ and $x = a^n y$ for some

$0 \neq y \in R$. Since $a^{2n}y = a^n x = 0$, $r(a^n) \subseteq r(a^{2n})$, this shows that the chain $r(a^n) \subset r(a^{2n}) \subset r(a^{3n}) \subset \Lambda \subset r(a^{kn}) \subset \Lambda$ is not stationary, so the descending chain of left ideal $Ra \supseteq Ra^{2n} \supseteq Ra^{3n} \supseteq \Lambda$ is not stationary, a contradiction of $\pi -$regularity of R. $(3) \Rightarrow (2)$. Is obvious.

Lemma 3.3. Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. If $a^n \in RP$ for some positive integer n, then the chains (3.4) and (3.5) are stationary.

Proof. Assume that $a^n \in RP$ for some positive integer n. Since RP is ideal of R, $a^n R + RP = RP$ and so $RP \subseteq a^{n+1}R + RP \subseteq a^n R + RP = RP$. This shows that the chain (3.4) is stationary. Similarly we can proof that the chain (3.5) is stationary.

Lemma 3.4. Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. If $a^n \in P$ for some positive integer n, then the chains (3.3), (3.4) and (3.5) are stationary.

Proof. Assume that $a^n \in P$ for some positive integer n. Then $a^n R + P = P$ and

$$P \subseteq a^{n+1}R + P \subseteq a^n R + P = P$$

This shows that the chain (3.3) is stationary. Since $a^n \in P \subseteq RP$, so by Lemma 3.3 the chains (3.4) and (3.5) are stationary.

Lemma 3.5. Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. Then the following statements are equivalent:

1 – The chain (3.3) is stationary.

2 – $R = aR + r(a^n)$ for some positive integer n.

3 – $R = a^n R + r_P(a^n)$ for some positive integer n.

Proof. $(1) \Rightarrow (2)$. Assume that the chain (3.3) is stationary. Then $a^n R + P = a^{n+1}R + P$ for some positive integer n. So $a^n = a^{n+1}x + p_0$ for some $x \in R$ and $p_0 \in P$. Thus $a^n(1 - ax) = p_0 \in P$, this shows that $(1 - ax) \in r_P(a^n)$ and so $R = aR + r_P(a^n)$.

$(2) \Rightarrow (1)$ .Suppose that $R = aR + r_P(a^n)$ for some positive integer n. Then $1 = ax + y$ for some $x \in R$ and $y \in r_P(a^n)$, since $a^n y \in P$, $a^n = a^{n+1}x + a^n y \in a^{n+1}R + P$. This shows that $a^n R + P = a^{n+1}R + P$. Thus, the chain (3.3) is stationary.

$(1) \Rightarrow (3)$. Assume that the chain (3.3) is stationary. Then

$$a^n R + P = a^{n+1}R + P = a^{2n}R + P$$

for some positive integer n. So $a^n = a^{2n}x + p_0$ for some $x \in R$, $p_0 \in P$ and $a^n(1 - a^n x) = p_0 \in P$, this shows that $1 - a^n x \in r_P(a^n)$ and so $R = a^n R + r_P(a^n)$.

$(3) \Rightarrow (2)$. Is obvious.

Note that in Lemma 3.5 and for $P = 0$ we derive Lemma 3.1.

Theorem 3.6. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:

1 – The chain (3.4) is stationary for every $a \in R$.

2 – The chain (3.5) is stationary for every $a \in R$.

Proof. Let $a \in R$, then by assumption there is a positive integer n such that

$$a^n R + RP = a^k R + RP \qquad (1)$$

for every integer $k \geq n$. If $a^t \in RP$ for some positive integer t, then by Lemma 3.3 the chain (3.5) is stationary. Suppose that $a^t \notin RP$ for every positive integer t. By (1) there are $x \in R$, $p_0 \in RP$ such that $a^n = a^{n+1}x + p_0$ and $x \notin RP$, hence if $x \in RP$, then $a^n = a^{n+1}x + p_0 \in RRP \subseteq RP$ a contradiction. Also, it is easy to see that $x^t \notin RP$ and $a^n x^t \notin RP$ for every positive integer t. By assumption the chain

$$xR + RP \supseteq x^2 R + RP \supseteq x^3 R + RP \supseteq \Lambda$$

is stationary, so there is a positive integer m such that $x^m R + RP = x^k R + RP$ for every integer $k \geq m$. So $x^m = x^{m+1}y + p_1$ where $y \in R$, $p_1 \in RP$ and $y \notin RP$, hence if $y \in RP$, then $x^m = x^{m+1}y + p_1 \in RRP \subseteq RP$ a contradiction. Let $\alpha = a^{n+m}$, $\beta = x^{n+m}$ and $\gamma = y^{n+m}$, then $\alpha = \alpha^2\beta + p'$ for some $p' \in RP$, hence

$$\alpha = a^{n+m} = a^m a^n = a^m(a^{n+1}x + p_0) = a^{m+1}a^n x + a^m p_0 =$$
$$= a^{m+1}(a^{n+1}x + p_0)x + a^m p_0 = a^{m+2}a^n x^2 + a^{m+1}p_0 x + a^m p_0 = \Lambda =$$
$$= a^{2(n+m)}x^{n+m} + \sum_{t=0}^{(n+m)-1} a^{m+t} p_0 x^t$$

Let $p' = \sum_{t=0}^{(n+m)-1} a^{m+t} p_0 x^t \in RP$, then $\alpha = \alpha^2\beta + p'$. Similarly we obtain $\beta = \beta^2\gamma + p''$ where $p'' = \sum_{t=0}^{(n+m)-1} x^{m+t} p_1 y^t \in RP$.

$$\alpha = \alpha^2\beta + p', \qquad \beta = \beta^2\gamma + p'', \qquad (2)$$

Also, by (2) found

$$\alpha\beta\gamma = (\alpha^2\beta + p')\beta\gamma = \alpha^2\beta^2\gamma + p'\beta\gamma =$$
$$= \alpha^2(\beta - p'') + p'\beta\gamma = \alpha^2\beta\alpha^2 p'' + p'\beta\gamma =$$
$$= \alpha - p' - \alpha^2 p'' + p'\beta\gamma$$

for $p_2 = -p' - \alpha^2 p'' + p'\beta\gamma \in RP$ implies that

$$\alpha\beta\gamma = \alpha + p_2 \qquad (3)$$

In addition to, by (2) and (3)

$$\alpha\gamma = \alpha^2 + p_3, \qquad \alpha(\gamma - \alpha) = p_3 \qquad (4)$$

where $p_3 = \alpha p_2 + p'\gamma \in RP$. So by (2) and (4)

$$\beta^2(\gamma - \alpha)^2 = \beta^2\gamma^2 - \beta^2\gamma\alpha - \beta^2\alpha\gamma + \beta^2\alpha^2 =$$
$$= \beta^2\gamma(\gamma - \alpha) - \beta^2\alpha(\gamma - \alpha) = \beta^2\gamma(\gamma - \alpha) - \beta^2 p_3 =$$
$$= (\beta - p'')(\gamma - \alpha) - \beta^2 p_3 = \beta(\gamma - \alpha) + p_4$$

where $p_4 = -p''(\gamma-\alpha) - \beta^2 p_3 \in RP$. i.e.,

$$\beta^2(\gamma-\alpha)^2 = \beta(\gamma-\alpha) + p_4 \qquad (5)$$

Thus by (5)

$$\beta^s(\gamma-\alpha)^s = \beta(\gamma-\alpha) + p_5, \quad s \geq 2 \qquad (6)$$

where $p_5 = \sum_{\lambda=0}^{s-2} \beta\lambda p_4(\gamma-\alpha)^\lambda \in RP$. On the other hand, by (2) and (3)

$$\alpha - \alpha\beta^2\alpha^2 = (\alpha\beta\gamma - p_2) - \alpha\beta\alpha + \alpha\beta\alpha - \alpha\beta^2\alpha^2 =$$
$$= \alpha\beta(\gamma-\alpha) + (\alpha\beta - \alpha\beta^2\alpha)\alpha - p_2 =$$
$$= \alpha\beta(\gamma-\alpha) + [\alpha(\beta^2\gamma + p'') - \alpha\beta^2\alpha]\alpha - p_2 =$$
$$= \alpha\beta(\gamma-\alpha) + \alpha\beta^2(\gamma-\alpha)\alpha + \alpha p'' - p_2$$

for $p_6 = -\alpha p''\alpha - p_2 \in RP$ implies

$$\alpha - \alpha\beta^2\alpha^2 = \alpha\beta(\gamma-\alpha) + \alpha\beta^2(\gamma-\alpha)\alpha + p_6 \qquad (7)$$

Since $\gamma - \alpha \in R$, then by assumption the chain

$$(\gamma-\alpha)R + RP \supseteq (\gamma-\alpha)^2 R + RP \supseteq \Lambda$$

is stationary. So there is a positive integer t such that

$$(\gamma-\alpha)^t = (\gamma-\alpha)^{t+1} z + p'''$$

for some $z \in R$, $p''' \in RP$. So by (6)

$$\alpha - \alpha\beta^2\alpha^2 = \alpha\beta(\gamma-\alpha) + \alpha\beta^2(\gamma-\alpha)\alpha + p_6 =$$
$$= \alpha\beta^{t+1}(\gamma-\alpha)^{t+1} + \alpha\beta^{t+1}(\gamma-\alpha)^t\alpha - \alpha p_5 - \alpha\beta p_5\alpha + p_6 =$$
$$= \alpha\beta^{t+1}(\gamma-\alpha)^t\gamma - \alpha p_5 - \alpha\beta p_5\alpha + p_6 =$$

for $p_7 = -\alpha p_5 - \alpha\beta p_5\alpha + p_6 \in RP$ implies that

$$\alpha - \alpha\beta^2\alpha^2 = \alpha\beta^{t+1}(\gamma-\alpha)^t\gamma + p_7$$
$$\alpha - \alpha\beta^2\alpha^2 = \alpha\beta^{t+1}(\gamma-\alpha)^{t+1} z\gamma + p_8 \qquad (8)$$

where $p_8 = \alpha\beta^{t+1} p'''\gamma + p_7 \in RP$ and so

$$\alpha - \alpha\beta^2\alpha^2 = \alpha\beta^{t+1}(\gamma-\alpha)[(\gamma-\alpha)^{t+1} z + p''']z\gamma + p_8$$
$$\alpha - \alpha\beta^2\alpha^2 = \alpha\beta^{t+1}(\gamma-\alpha)^{t+2} z^2\gamma + p_9 \qquad (9)$$

where $p_9 = \alpha\beta^{t+1}(\gamma-\alpha)p''' z\gamma + p_8 \in RP$. By (6) implies that

$$\alpha - \alpha\beta^2\alpha^2 = \alpha[\beta^{t+1}(\gamma-\alpha)^{t+1}](\gamma-\alpha)z^2\gamma + p_9 =$$
$$= \alpha[\beta(\gamma-\alpha) + p_5](\gamma-\alpha)z^2\gamma + p_9$$
$$\alpha - \alpha\beta^2\alpha^2 = (\alpha\beta\gamma - \alpha\beta\alpha)(\gamma-\alpha)z^2\gamma + p_{10} \qquad (10)$$

where $p_{10} = \alpha p_5(\gamma-\alpha)z^2\gamma + p_9 \in RP$. By (3) found

$$\alpha - \alpha\beta^2\alpha^2 = (\alpha + p_2 - \alpha\beta\alpha)(\gamma-\alpha)z^2\gamma + p_{10}$$
$$\alpha - \alpha\beta^2\alpha^2 = (\alpha - \alpha\beta\alpha)(\gamma-\alpha)z^2\gamma + p_{11}$$

where $p_{11} = p_2(\gamma-\alpha)z^2\gamma + p_{10} \in RP$, and by (4) implies that

$$\alpha - \alpha\beta^2\alpha^2 = \alpha(\gamma-\alpha)z^2\gamma - \alpha\beta\alpha(\gamma-\alpha)z^2\gamma + p_{11} =$$

$= p_3 z^2 \gamma - \alpha\beta p_3 z^2 \gamma + p_{11} \in RP$

$\alpha - \alpha\beta^2\alpha^2 = p_{12}$ (11)

where $p_{12} = p_3 z^2 \gamma - \alpha\beta p_3 z^2 \gamma + p_{11} \in RP$. Since $\alpha = a^{n+m}$ follows that

$a^{n+m} = \alpha\beta^2 a^{2(n+m)} + p_{12}$

and for $\lambda = n+m$ found

$a^{\lambda} = (\alpha\beta^2 a^{\lambda-1})a^{\lambda+1} + p_{12} \in Ra^{\lambda+1} + RP$

This shows that the chain (3.5) is stationary for every $a \in R$.

$(2) \Rightarrow (1)$ The proof is analogous to the proof of $(1) \Rightarrow (2)$.

Let R be a ring and $P \neq R$ be a right ideal of R. It is easy to see that for every $a \in R$:

- The set $r_{RP}(a) = \{x : x \in R;\ ax \in RP\}$ is a right ideal of R.
- The set $\lambda_{RP}(a) = \{x : x \in R;\ xa \in RP\}$ is a left ideal of R.

We say that a ring R is strongly $\pi-$regular relative to ideal RP if the chain (3.4) is stationary for every $a \in R$.

Theorem 3.7. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:

1 – A ring R is strongly regular relative to ideal RP.

2 – For every $a \in R$ the chain (3.4) is stationary.

3 – For every $a \in R$ the chain (3.5) is stationary.

4 – For every $a \in R$ there is $b \in R$ such that $a^n = a^{n+1}b + z$ for some positive integer n and $z \in RP$.

5 – For every $a \in R$ there is $c \in R$ such that $a^m = ca^{m+1} + y$ for some positive integer m and $y \in RP$.

6 – For every $a \in R$ there is a positive integer n such that $R = a^n R + r_{RP}(a^n)$.

7 – For every $a \in R$ there is a positive integer n such that $R = aR + r_{RP}(a^n)$.

8 – For every $a \in R$ there is a positive integer m such that $R = Ra^m + \lambda_{RP}(a^m)$.

9 – For every $a \in R$ there is a positive integer m such that $R = Ra + \lambda_{RP}(a^m)$.

10 – For every $a \in R$ there is a positive integer t such that

$a^t \in (a^{t+k}R + RP) \cap (Ra^{t+k} + RP)$

for all positive integer k.

Proof. $(1) \Leftrightarrow (2)$. By definition. $(2) \Leftrightarrow (3)$. By theorem 3.6. $(2) \Leftrightarrow (4)$ and $(3) \Leftrightarrow (5)$ are directly verified. $(4) \Rightarrow (6)$. Let $a \in R$, then there is $b \in R$ such that $a^n = a^{n+1}b + z$

for some positive integer n and $z \in RP$, so

$a^n = a^{n+2}b^2 + azb = \Lambda = a^{2n}b^n + p_0$

where $p_0 = \sum_{t=0}^{n-1} a^t z b^t \in RP$, thus $a^n(1-a^n b^n) = p_0 \in RP$ this shows that
$1-a^n b^n \in r_{RP}(a^n)$
thus $R = a^n R + r_{RP}(a^n)$. $(6) \Rightarrow (7)$. Is obvious.

$(7) \Rightarrow (4)$. Let $a \in R$, then $R = aR + r_{RP}(a^n)$ for some positive integer n. So $1 = ax + y$ for some $x \in R$ and $y \in r_{RP}(a^n)$, thus $a^n = a^{n+1}x + a^n y$ and $a^n y \in RP$.

$(5) \Rightarrow (8) \Rightarrow (9) \Rightarrow (5)$. The proof is analogous to the proof of $(4) \Rightarrow (6) \Rightarrow (7) \Rightarrow (4)$.

$(10) \Rightarrow (4)$ and $(10) \Rightarrow (5)$ are obvious. $((4)+(5)) \Rightarrow (10)$. Let $a \in R$. By (4) and (5) there are positive integers n and m such that $a^n = a^{n+1}b + p_0$ and $a^m = ca^{m+1} + p_1$ where $b, c \in R$ and $p_0, p_1 \in RP$. Then $a^{n+m} = a^{n+m+1}b + a^m p_0$ where $a^m p_0 \in RP$ and $a^{n+m} = ca^{n+m+1} + p_1 a^n$ where $p_1 a^n \in RP$.

So $a^{n+m} \in a^{n+m+k}R + RP$ and $a^{n+m} \in Ra^{n+m+k} + RP$ for every positive integer k. For $t = n+m$ implies that $a^t \in (a^{t+k}R + RP) \cap (Ra^{t+k} + RP)$.

Corollary 3.8. Let R be a ring and $P \neq R$ be a right ideal of R. If for every $a \in R$ the chain (3.3) is stationary, then for every $a \in R$ the chains (3.4) and (3.5) are stationary.

4. $P-$Potent Rings.

Recall that a ring R is an I_0-ring [6], if every principal left (right) ideal of R not contained in $J(R)$ contains a nonzero idempotent. We say that a ring R is a $P-$potent ring or I_0-ring relative to a right ideal $P \neq R$ of R, if for any $a \in R$ there exists $b \in R$ such that $bab - b \in P$ and $baP \subseteq P$. In the following we provide characterization of this rings in terms of principal left ideals:

Theorem 4.1. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:

1 – A ring R is $P-$potent.

2 – Every left ideal of R contains a $P-$idempotent element.

3 – For every $a \in R$, Ra contains a $P-$idempotent element.

Proof. $(1) \Rightarrow (2)$. Let A be a left ideal of R and $a \in A$, by assumption there exists $x \in R$ such that $xax - x \in P$ and $xaP \subseteq P$. Let we define $e = xa$, then

$$e^2 - e = xaxa - xa = (xax - x)a \in PR \subseteq P$$

and $eP = xaP \subseteq P$, so e is $P-$idempotent and $e = xa \in RA \subseteq A$. $(2) \Rightarrow (3)$. Is obvious.

$(3) \Rightarrow (1)$. Let $a \in R$, by assumption there exists $P-$idempotent $e \in R$ such that $e \in Ra$. Then $e^2 - e \in P$, $eP \subseteq P$ and $e = xa$ for some $x \in R$, so $e^2 = e + p_0$ where $p_0 \in P$. Let we define $b = xax$, then

$$bab = (xax)a(xax) = e^2 ex = (e + p_0)ex = e^2 x + p_0 ex =$$

$=(e+p_0)x+p_0ex=ex+p_0x+p_0ex==xax+p_1=b+p_1$

where $p_1=p_0x+p_0ex\in PR\subseteq P$. Thus $bab-b=p_1\in P$ and

$baP=xaxaP=e(eP)\subseteq eP\subseteq P$

this shows that R is a $P-$potent ring.

In the following we prove a new characterization of the $P-$potent ring in terms of $P-$annihilator:

Theorem 4.2. Let R be a ring and $P\neq R$ be a right ideal of R. Then the following statements are equivalent:

1- A ring R is $P-$potent.
2- For every $a\in R$ there exists $b\in R$ such that $bR+P=r_P(1-ba)$.
3- For every $a\in R$ there exists $b\in R$ such that $(1-ba)R+P=eR+P$ for some $P-$idempotent $e\in R$.

Proof. $(1)\Rightarrow(2)$. Let $a\in R$, then $bab-b\in P$ and $baP\subseteq P$ for some $b\in R$. So $(1-ba)b\in P$, thus $b\in r_P(1-ba)$ and hence $r_P(1-ba)$ is a right ideal in R, $bR\subseteq r_P(1-ba)$. Also, for any $t\in P$, $(1-ba)t=t-bat\in P+baP\subseteq P$, so $t\in r_P(1-ba)$ i.e. $P\subseteq r_P(1-ba)$, thus $bR+P\subseteq r_P(1-ba)$. On the other hand, let $y\in r_P(1-ba)$, then $(1-ba)y\in P$, so $y=bay+p_0\in bR+P$ for some $p_0\in P$, therefore $r_P(1-ba)\subseteq bR+P$. This proves (2).

$(2)\Rightarrow(3)$. Since $b\in bR\subseteq r_P(1-ba)$, $b-bab\in P$. Also, since for any $t\in r_P(1-ba)$, $(1-ba)t\in P$ so $t-bat=t_0\in P$ and $bat=t-t_0\in P$, i.e. $baP\subseteq P$. Let $e=ba\in R$, then e is $P-$idempotent and $(1-ba)R+P=(1-e)R+P$ where $1-e\in R$ is $P-$idempotent.

$(3)\Rightarrow(1)$. Let $a\in R$, by assumption there exists $b\in R$ such that $(1-ba)R+P=eR+P$ for some $P-$idempotent $e\in R$. Then $e^2-e\in P$, $eP\subseteq P$ so $e^2=e+p_0$ for some $p_0\in P$. Since $1-ba\in eR+P$,

$$1-ba=ex+p_1 \qquad (1)$$

for some $p_1\in P$ and so

$e(1-ba)=e^2x+ep_1=(e+p_0)x+ep_1=ex+p_0x+ep_1$

by (1) we have

$e(1-ba)=(1-ba)-p_1+p_0x+ep_1=1-ba+p_2$

where $p_2=-p_1+p_0x+ep_1\in P+PR+eP\subseteq P$. Thus $ba-eba=(1-e)+p_2$ and so

$$(1-e)ba=(1-e)+p_2 \qquad (2)$$

$(1-e)ba(1-e)=(1-e)^2+p_2(1-e)$

since $1-e$ is $P-$idempotent, $(1-e)^2-(1-e)\in P$ and $(1-e)P\subseteq P$, so

$(1-e)^2=(1-e)+p_3$

and so

$$(1-e)ba(1-e)b=(1-e)b+p_3b+p_2(1-e)b=(1-e)b+p_4$$

where $p_4=p_3b+p_2(1-e)b\in PR+PR\subseteq P$. Let we define $d=(1-e)b\in R$, then $dad-d=p_4\in P$ and by (2) we have

$$daP=(1-e)baP=((1-e)+p_2)P\subseteq(1-e)P+p_2P\subseteq P$$

This shows that R is $P-$potent.

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