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Predicting the Winner in the Game of Nim

Khaled Sliman Aloklah

Faculty of Science || Albaath University || Homs || Syria

Abstract: Game Theory is defined as a means of mathematical analysis when interests collide with each other to reach the best possible decision-making options taking into consideration the given circumstances to get the desired results. Even though Game Theory is related to well-known games such as checkers, XO, and poker. In fact, it is associated with more serious problems pertaining to sociology, economics, politics, military sciences. Game Theory includes several sorts of games like Combinatorial Games.

This Paper gives the reader a detailed outlook of a special Combinatorial game called Nim Game. Before the start of the game we will develop a strategy to determine the winner in advance with the help of some basic mathematical concepts like Minimum Excluded (MEX), Grundy Numbers and The XOR Number.

Keywords: Combinatorial games, Nim game, Minimum Excluded (MEX), Grundy Numbers, XOR Number, Expect the winner.

توقُّع الفائز في لُعبة نِيم

خالد سليمان العكله

كلية العلوم || جامعة البعث || حمص || سوريا

الملخص: نظرية الألعاب تعرّف بأنها وسيلة من وسائل التحليل الرياضي لحالات تضارب المصالح، للوصول إلى أفضل الخيارات المكنة لاتخاذ القرار في ظل الظروف المعطاة لأجل الحصول على النتائج المرغوبة. بالرغم من ارتباط نظرية الألعاب بالتسالي المعروفة كلعبة الداما، إكس أو، البوكر، إلا أنها تخوض في معضلات أكثر جدية تتعلق بعلم الاجتماع، والاقتصاد، والسياسة، بالإضافة إلى العلوم العسكرية، كما ويندرج تحت نظرية الألعاب عدة أنواع من الألعاب منها الألعاب التآلفية.

يعطي هذا البحث للقارئ نظرة تفصيلية لإحدى أهم الألعاب التآلفية والتي تسمى لعبة (نيم)، قبل بدء اللعبة سنضع استراتيجية لتحديد الفائز مقدماً بمساعدة بعض المفاهيم الرياضية الأساسية مثل: الحد الأدنى المستبعد، أعداد غراندي، العدد XOR.

الكلمات المفتاحية: الألعاب المدمجة(التآلفية)، ألعاب نيم، الحد الأدنى المستبعد، أعداد غراندي، أعداد إكس أور، الرابح المتوقع.

The Problem of the Study:

The important application in this Paper was to infer a strategy to predict the winner in a distinguished game called Nim before starting the game, A number of concepts have been introduced like Minimum Excluded and Grundy Numbers.

The importance of the Study:

The importance of this study comes from the study of a complex and important type of games, the theory of games is very important in our working life and intervene in all aspects of life, what we have done is a small part of the combinatorial games, which in turn fall under the theory of games, the science known.

Research Method:

In this study, the scientific descriptive method was used, and the deductive method was used based on several previous researchers that studied a similar type of games.

1. Historical Introduction

Nim forms the foundation of the mathematical study of two-player strategy games [1,2]. Nim, a game with a complete mathematical theory, Charles L. Bouton [3] provided a solution to the game of Nim, essentially founding the field of Combinatorial Game Theory. Since Bouton's discovery, many extensions or variants of Nim have been explored. Some variations that come to mind are Wythoff's Game, Poker Nim and Kayles. These variations often yield winning strategies that bare little resemblance to that of Nim. There are two kinds of combinatorial game, Partisan and Impartial Games [4]. The difference between them is that in Impartial Games all the possible moves from any position in the game are the same for all players, whereas in Partisan Games the moves for all the players are not the same like Chess.

In this Paper, we explore a new strategy to determine the winner before starting the game, as we know Nim is a combinatorial game and it is an impartial Game which means the two players have the same options.

2. Nim Game

"Combinatorial Game Theory" describes the study of sequential games with perfect information. When playing a "Combinatorial Game", all players know all the possible outcomes from a given position with no randomness [5]. In general, the game is defined with the following attributes:

(A) There are two players.

(B) There is always finite number of positions, in addition to a starting position.

The game is played by the following rules:

(C) players alternate making moves.

(D) Both players have access to all information always.

(E) There is no randomness to moves made, such as rolling a dice.

(F) A player loses when he/she can no longer make any legal moves.

(G) The game ends when the ending condition is met.

If we have a game G and this game includes options for player (1) named G_1 and options for player (2) named G_2 .

Now, we will try to explain the manner of dealing with the Games , as well as showing its rules.

$G = \{G_1|G_2\}$ or $G = (G_1, G_2)$

Pretty much the same in Nim Game, Nim is an ancient game with several variations [6,7]. Here's one: Two players take turns removing marbles from a pile. On each turn, the player must remove at least one but no more than half of the remaining marbles. The player who is forced to remove the last marble loses for example see Figure 1.

Figure (1) Type (1) of Nim Game.

There is another type, for example we have several piles of stones, and you have to take a limited number of stones from a specific pile and this is what we will study in this Paper, see Figure 2.

Figure (2) Type (2) of Nim Game.

Definition 2.1. A Zero-game

A Zero-game is a game that scores $\{ \mid \} = 0$, or in other words, the player who moves next lose, assuming all moves made are optimal [8,9].

For Nim, the simplest form of a Zero game equates to an empty board at the beginning. Thus, it is obvious the first player to move has no legal move, therefore loses. In other words, an endgame, denoted by $\emptyset$, is a game where no legal move is available to the current player.

3. How to win

Now we will have some concepts that will help us reach the strategy that will determine the winner in the game of Nim before the start of the game.

Definition 3.1. A Minimum Excluded (MEX)

This is an operation that we need to perform on a set [10,11], so if we have a set of numbers

$$\mathrm{S} = \{0,1,3,8,12\}$$

which are non-negative and if we perform this operation on this set:

$$\mathrm{MEX(S)} = 2$$

we get the smallest number which does not exist in this set, the reason why you might want to use this is because,

- S gives us the set of moves that we can play
- MEX tell us the smallest move and the first move that we can never play.

That might be useful information to have and therefore that is the operation we keep in mind. Using this very operation, we can calculate something called Grundy Numbers.

Definition 3.2. Grundy Numbers

If we have N piles of coins with different coins in each one, a single Grundy Number can define the entire game state, every impartial game has this Grundy Number [12,13]. The way to calculate it, is to take the MEX of the set, if the set is empty, that is the basic condition and MEX of anything else is equal to the way that we have defined above [14].

$$\text{Grundy Number} = \text{MEX}(\text{S}), \text{MEX}(\emptyset) = 0$$

Grundy Number is MEX of set of moves that we can play.

Grundy (position) = MEX (set of possible position).

Example3.1

If we have a pile of six coins and the only kinds of moves that we can make is either take one or two or three coins. That termination condition is used if we have no move coins to take and if we have no move possible, then we lose the game. So, we need to play optimally from six coins such that our opponent loses

$$\text{Grundy (6)} = \text{MEX}\left(\text{Grundy (5)}, \text{Grundy (4)}, \text{Grundy (3)}\right)$$
$$\text{Grundy (0)} = \text{MEX}(\emptyset) = 0$$
$$\text{Grundy (1)} = \text{MEX}\left(\text{Grundy (0)}\right) = \text{MEX}(0) = 1$$
$$\text{Grundy (2)} = \text{MEX}\left(\text{G (0)}, \text{G (1)}\right) = \text{MEX}(0{,}1) = 2$$
$$\text{Grundy (3)} = \text{MEX}(2{,}1{,}0) = 3$$
$$\text{Grundy (4)} = \text{MEX}\left(\text{G (3)}, \text{G (2)}, \text{G (1)}\right) = \text{MEX}(3{,}2{,}1) = 0$$
$$\text{Grundy (5)} = \text{MEX}\left(\text{G (4)}, \text{G (3)}, \text{G (2)}\right) = \text{MEX}(0{,}3{,}2) = 1$$
$$\text{Grundy (6)} = \text{MEX}\left(\text{G (5)}, \text{G (4)}, \text{G (3)}\right) = \text{MEX}(1{,}0{,}3) = 2$$

Definition 3.3. XOR Number and the winner

As we say, Grundy (position) = MEX (set of possible position), XOR deals with binary numbers [15,16], if we collected an odd number of units, we will get number one and if we collect an even number of units, we will get zero.

$$0 + 0 = 0, 1 + 0 = 1{,}0 + 1 = 1, 1 + 1 = 0$$
$$\text{XOR(set of possible position)} = \begin{bmatrix} 0\ losing \\ 1 = 0\ winning \end{bmatrix}$$

Example 3.2

The same previous example, if we have a pile of six coins and the only kinds of moves that we can make is either take one or two or three coins $1 \leq \text{coins} \leq 3$

$$\text{Result (6)} = \text{XOR}\left(\text{Grundy (5)}, \text{Grundy (4)}, \text{Grundy (3)}\right)$$

$= \text{XOR}\ (1,0,3) = 2$
$\text{Result}\ (5) = \text{XOR}\ \left(\text{Grundy}\ (4), \text{Grundy}\ (3), \text{Grundy}\ (2)\right)$
$= \text{XOR}\ (0,3,2) = 1$
$\text{Result}\ (4) = \text{XOR}\ \left(\text{Grundy}\ (3), \text{Grundy}\ (2), \text{Grundy}\ (1)\right)$
$= \text{XOR}\ (3,2,1) = 0$

$$\begin{array}{ccc} 0\,1 & 1\,0 & 1\,0 \\ 0\,0 & 0\,0 & 0\,1 \\ 1\,1 & 1\,1 & 1\,1 \\ \hline (10)_2 = 2 & (0\,1)_2 = 1 & (00)_2 = 0 \end{array}$$

Here is a summary of the above example in the table.1

Table (1) Determine the winner by the number of coins.

Number of coins	Winner or Loser
6	W
5	W
4	L

$$\text{Game (State)} \rightarrow \begin{array}{c} \text{Result}\ (S_1) \\ \text{Result}\ (S_2) \\ \vdots \\ \text{Result}\ (S_n) \end{array}$$

$$\text{Result (S)} = \text{XOR}\ (P \in S).$$

If any one of these results equals to zero, we could choose to move from the first state to this state, in the previous example the optimal is to move from six coins to four coins.

Example 3.3

We have 4 piles and we have 4, 6, 8 and 2 coins on each pile, the operation is division (the floor of division), for example if we choose the pile of six coins, we can move to a pile of 1 coin by dividing by 6, see Figure 3.

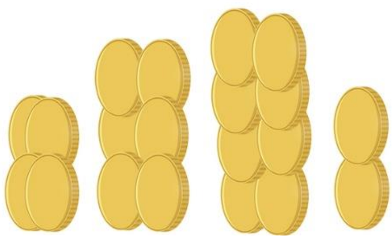

Figure (3) (4,6,8,2) Nim Game.

and we will find who will win.

$$\text{Result} = \text{XOR}\ \begin{pmatrix} \text{Grundy}\ (4), \text{Grundy}\ (6), \\ \text{Grundy}\ (8), \text{Grundy}\ (2) \end{pmatrix}$$

$$\text{Grundy}\ (X) = \text{MEX}(S)$$

$$\mathrm{G}\ (2) = \mathrm{MEX}\big(\mathrm{G}\ (1), \mathrm{G}\ (0), \mathrm{G}\ (0)\big) = \mathrm{MEX}(1{,}0{,}0) = 2$$
$$\mathrm{G}\ (4) = \mathrm{MEX}\big(\mathrm{G}\ (2), \mathrm{G}\ (1), \mathrm{G}\ (0)\big) = \mathrm{MEX}(2{,}1{,}0) = 3$$
$$\mathrm{G}\ (3) = \mathrm{MEX}\big(\mathrm{G}\ (0), \mathrm{G}\ (1), \mathrm{G}\ (1)\big) = \mathrm{MEX}(0{,}1{,}1) = 2$$
$$\mathrm{G}\ (6) = \mathrm{MEX}\big(\mathrm{G}\ (1), \mathrm{G}\ (2), \mathrm{G}\ (3)\big) = \mathrm{MEX}(1{,}2{,}2) = 0$$
$$\mathrm{G}\ (8) = \mathrm{MEX}\big(\mathrm{G}\ (1), \mathrm{G}\ (2), \mathrm{G}\ (4)\big) = \mathrm{MEX}(1{,}2{,}3) = 0$$
$$\mathrm{Result} = \mathrm{XOR}\ (3{,}0{,}0{,}2) = 1$$

$$\begin{array}{c} 1\ 0 \\ 0\ 0 \\ 0\ 0 \\ 1\ 1 \\ \hline (01)_2 = 1 \end{array}$$

This means that the person who is starting this game will win.

Now we need to make the game in a losing position, we need to make $\mathrm{XOR} = 1 \ \rightarrow \mathrm{XOR} = 0$, if we notice $\mathrm{XOR}\ (2{,}0{,}0{,}2) = 0$

This means we need to move from three to two which means to Move from 4 coins to 2 coins, see Figure 4.

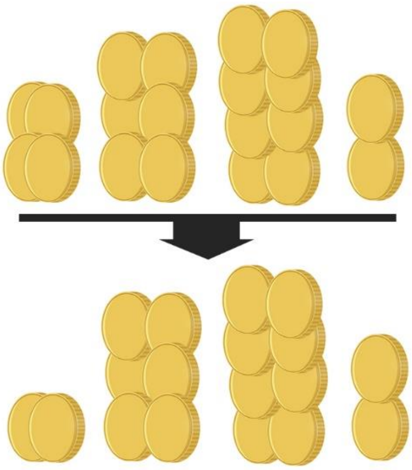

Figure (4) Moving to(2,6,8,2) Nim Game.

that is the optimal move to make this game in a losing position.

3. Results and discussion

we have used several mathematical concepts to arrive at a new way which we can predict who will be the winner in the end of the game of Nim.

4. Conclusion

In this Paper, we have utilized combinatorial mathematical operations and concepts, to define a new strategy to predict the winner before the start of an important game called "Nim Game".

We used Minimum Excluded (MEX), Grundy Numbers and The XOR Number.

5. Future Work

What happens if we considered the game with N piles or infinite number of piles.

$$\mathrm{Result} = \mathrm{XOR}\ (\mathrm{X}_1, \mathrm{X}_2, \cdots, \mathrm{X}_n) = ?$$

Only further work will tell how far Combinatorial Game theory can go.

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