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Piecewise Linear Discontinuous Petrov Galerkin Method for Time Fractional Diffusion Equations

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Abstract: We propose and analyze piecwise linear discontinuous Petrov-Galerkin method in time combined with a standard conforming finite element method in space for the numerical solution of time-fractional diffusion problems of order $0 < \mu < 1$. We prove the stability of the exact solution. The existence, uniqueness and stability of approximate solutions will be proved. We employ a non-uniform mesh based on concentrating the cells near the singularity. The advantage of employing a non-uniform mesh is improving the accuracy of the approximate solution. Numerical experiments indicate the error in $L\infty(0, T; L2(\Omega))$-norm is of order $k^{\min(\gamma(1-\mu),2)} + h^2$, where k denotes the maximum time steps and h is the maximum diameter of the elements of the (quasi-uniform) spatial mesh and $\gamma > 0$.

Keywords: Fractional Derivatives, Petrov-Galerkin Method, Finite Element Method, Stability.

طريقة بيتروف جاليركين الخطية المنفصلة لمعادلات الانتشار الكسرية للزمن

الدكتور / بشير صالح عبد الله

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المستخلص: المشتقات الكسرية توفر أداة لوصف الذاكرة والوراثة لمختلف المواد والعمليات. من مزايا المشتقات الكسرية أصبحت واضحة في نمذجة الخواص الميكانيكية والكهربائية في المواد الحقيقية، في وصف خصائص تدفق السوائل واللزوجة، في الفيزياء الكيميائية، في البصريات ومعالجة الإشارات، وفي العديد من المجالات الأخرى.

في هذا البحث نقترح طريقة بيتروف جاليركين الخطية المنفصلة للزمن مع طريقة العناصر المحددة للفضاء للحل العددي لمعادلات، لقد تم اثبات أن الحل المضبوط هو مستقر، وأيضا أن الحل التقريبي هو وحيد ومستفر. $0 < \mu < 1$. الإنتشارالكسرية الزمنية من الرتبة

بالإضافة إلى ذلك لقد قمنا بتوظيف تجزئة غير منتظمة تقوم على تركيز خلايا قرب التفرد. وهذا يؤدي إلى تحسين دقة الحل التقريبي.

وأخيرا لقد تم استخدام برنامج ماتلاب للحصول على النتائج العددية والتي تشير الى أن الخطأ بالنسبة للمعيار $L\infty(0, T; L2(\Omega))$ الحد الأقصى للقطر من الشبكة h الحد الأقصى للخطوات الزمنية ، و k حيث $k^{\min(\gamma(1-\mu),2)} + h^2$ هو من الرتبة $\gamma > 0$. الفضائية (شبه المنتظمة)لقيم

الكلمات المفتاحية: المشتقات الكسرية، طريقة بيتروف-جالركين، طريقة العناصر المحدودة، الثبات.

1- Introduction

In this paper, we propose and analyze the time-stepping discontinuous Petrov-Galerkin (DPG) method for a time-fractional diffusion model:

$$^{C}D^{1-\mu}v(x,t) - \Delta v(x,t) = f(x,t) \quad \text{in } \Omega \times (0,T], \quad (1a)$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T] \quad (1b)$$

$$v|_{t=0} = g(x) \quad \text{on } \partial\Omega \quad (1c)$$

where Ω is a convex polyhedral domain of R^n with n = 1, 2, 3 and $\frac{\partial v}{\partial n}$ is the derivative in the direction of the exterior unit normal n to $\Omega \times (0,T]$. Here $^{C}D^{1-\mu}$ $0 < \mu < 1$, is the time fractional Caputo derivative of v defined by

$$^{C}D^{1-\mu}v(x,t) = \int_0^t w_\mu(t-s)\, v'(x,s)ds \quad (2)$$

$$\text{where } w_\mu(t) = \frac{t^{\mu-1}}{\Gamma(\mu)}.$$

Problems of the form (1) refers to a partial differential equation that arise various scientific applications , specifically in the context of slow or a nomalous sub-diffusion [10, 15, 17, 18, 25, 27, 28]. This type of diffusion in systems is not characterized by a standard diffusive process and can be influenced by the entire history of the density gradient. Several authors have considered numerical methods for (1), see [7, 8, 9, 26, 29, 33 ,38] and references therein. Moreover, various numerical methods have been applied for the following alternative

representation of the fractional subdiffusion problem (1):

$$v'(x,t) - D^{1-\mu}\Delta v(x,t) = f(x,t) \qquad \text{in } \Omega \times (0,T], \quad (3)$$

where $D^{1-\mu}$ is a Riemann–Liouville fractional derivative defined by

$$D^{1-\mu}v(x,t) = \frac{\partial}{\partial t}\int_0^t w(t-s)v(x,s)ds$$

see for example, [2, 3, 4, 5, 6, 12, 14, 16, 20, 21, 22, 23, 24, 30, 31, 34, 35,37] and the references therein. Practically, the two representations are different ways of writing the same equation as they are equivalent under reasonable assumptions on the initial data, see [32]. However, the numerical methods obtained for each representation are formally different.

The main objective of the paper is to establish the well-posedness of the discontinuous

Petrov-Galerkin method combined with a standard conforming finite element method (DPGFE) method. In the analysis presented in the paper, we utilize the positivity, coercivity continuity

properties of the fractional time derivative operator in an nonstandard manner. This will

play a crucial role to establish the existence, uniqueness and stability.

We also present implementation and numerical results which indicate the error in $L_\infty(0, T; L_2(\Omega))$--norm of order $k^{\min(\gamma(1-\beta),2)} + h^2$, where k denotes the maximum time steps and h is the maximum diameter of the elements of the (quasi-uniform) spatial mesh and $\gamma > 0$.

The outline of the paper is as follows. In section 2, we define some definitions and theorems used in the next sections. Section 3 proves the stability of the exact solution.

Section 4 introduces a fully discrete DPG-FE scheme in addition to some notations. In Section 5, we prove our main results regarding the well-posedness of the approximate solutions. In Section 6, we implement and present numerical example.

2- Preliminaries

Definition 2.1. [10]: If $\mu > 0$, then I^{μ} is the Riemann–Liouville fractional integral defined by

$$I^{\mu}v(t) = \int_0^t \omega_{\mu}(t-s)v(s)ds \text{ with } \omega_{\mu}(t) = \frac{t^{\mu-1}}{\Gamma(\mu)}$$

Definition 2.2. [10]: $^{c}D^{\mu}$, $n-1 < \mu \le n$, is the fractional Caputo derivative definedby

$$D^{\mu}v(x,t) = I^{n-\mu}D^{n}v(t) = \int_0^t \omega_{\mu}(t-s)v^{(n)}(s)ds \quad (4)$$

Lemma 2.3 ([1],**Discrete Gronwall's Inequality**) Let $\{c_j\}_{j=1}^{n}$ and $\{d_j\}_{j=1}^{n}$ be sequences of non-negative numbers with $d_1 \le d_2 \le, \dots, \le d_n$. Assume that, for $a \ge 0$ and weights $(k_1, \dots, k_{n-1}) \in R^{n-1^+}$, $c_1 \le d_1$, $c_j \le d_j + a\sum_{i=1}^{j-1}(k_i\, c_i)$, $j = 2, \dots, n$. Then $c_j \le d_j \exp\left(C\sum_{i=1}^{j-1} k_i \right)$, $j = 2, \dots, n$, where R^+ is the set of positive real numbers.

Lemma 2.4: [11] For $1 \le j \le n$, and the interval I_j ,let $u_{|Ij}, v_{|Ij} \in H^1\left(Ij, L2(\Omega)\right) \cap C\left(Ij, L2(\Omega)\right)$. there holds

1. If $\int_0^{t_n}\langle {}^{C}D^{1-\mu}u(s), u'(s)\rangle\, ds$ and $\max_{j=1}\|u_j\|$= 0 , then u = 0
2. The continuity property: for any a> 0

$$\int_0^{t_n}\langle {}^{C}D^{1-\mu}u(s), v(s))\rangle\, ds \le$$

$$\frac{a}{2c_\mu^{\,2}}\int_0^{t}\langle {}^{C}D^{1-\mu}u(s), u'(s))\rangle\, ds + \frac{1}{2a}\int_0^{t}\langle {}^{C}D^{1-\mu}v(s), v'(s))\rangle\, ds \quad \text{with } c\ \mu = \cos(\ \mu\pi/2)$$

3. The coercivity property: $c_\mu\int_0^{t_n}\left\| cD^{1-\frac{\mu}{2}}u(t)\right\|^2 \mathrm{dt} \le \int_0^{t}\langle {}^{C}D^{1-\mu}u(s), u'(s)\rangle\, ds$ (5)

where $\| u \| = \| u \|_{L_2}$.

Lemma 2.5. [10] If $\alpha > 0$ and $\mu > 0$, then

$$I^{\alpha+\mu}v = I^{\alpha}\, I^{\mu}v \tag{6}$$

is satisfied at almost every point v ∈ [0,T] for u ∈ $L_p(0,T)$, 1 ≤ p ≤ ∞.

Lemma 2.6. *If* $0 < \mu < 1$*, and* $v \in H^1(0,T)$*, then we have*

$$\|v(t)\|^2 \le \|v(0)\|^2 + \frac{2T^{1-\mu}}{(1-\mu)\Gamma^2(1-\frac{\mu}{2})}\int_0^{t_n}\left\|{}^{C}D^{1-\frac{\mu}{2}}v(t)\right\|^2 dt \tag{7}$$

Proof. Since $v \in H^1(0,T)$, then we have

$v(t) = v(0) + Iv'(t)$

and writing $Iv'(t)$ as $Iv'(t) = I^{1-\frac{\mu}{2}+\frac{\mu}{2}}v'(t)$

and using (6) in lemma 2.5 and Cauchy Schwarz inequality, then

$$\|v(t)\| \le \|v(0)\| + \left\| I^{\frac{\mu}{2}}{}^{C}D^{1-\frac{\mu}{2}}v(t)\right\|$$

$$\le \|v(0)\| + \int_0^{t}\frac{(t-s)^{\frac{-\mu}{2}}}{\Gamma^2\left(1-\frac{\mu}{2}\right)}\left\|{}^{C}D^{1-\frac{\mu}{2}}v(s)\right\| ds \tag{8}$$

$$\le \|v(0)\| + \left(\int_0^{t}\frac{(t-s)^{-\mu}}{\Gamma^2\left(1-\frac{\mu}{2}\right)} ds\right)^{\frac{1}{2}}\left(\int_0^{t}\left\|{}^{C}D^{1-\frac{\mu}{2}}v(s)\right\|^2 ds\right)^{\frac{1}{2}} \tag{9}$$

$$\le \|v(0)\| + \left(\frac{t^{1-\mu}}{(1-\ \mu)\Gamma^2\left(1-\frac{\mu}{2}\right)}\right)^{\frac{1}{2}}\left(\int_0^{t}\left\|{}^{C}D^{1-\frac{\mu}{2}}v(s)\right\|^2 ds\right)^{\frac{1}{2}} \tag{10}$$

$$\le \|v(0)\| + \left(\frac{T^{1-\mu}}{(1-\ \mu)\Gamma^2\left(1-\frac{\mu}{2}\right)}\right)^{\frac{1}{2}}\left(\int_0^{T}\left\|{}^{C}D^{1-\frac{\mu}{2}}v(s)\right\|^2 ds\right)^{\frac{1}{2}} \tag{11}$$

Squaring both sides of the inequality (11) and utilizing the geometric-arithmetic mean in equality , then we obtain the desired result.

Inequality 2.7:[36](**Green's Inequality**) Let $u \in C^2$ and $v \in C^1$, then

$$\int_\Omega \nabla u\nabla v = \int_\Gamma \frac{\partial \mathrm{u}}{\partial n}\, \mathrm{v}\ ds - \int_\Omega \Delta u\ v\ dx$$

where $\frac{\partial \mathrm{u}}{\partial n} = n.\nabla u$ is the exterior normal derivative of u on Γ.

3- Stability of The Continuous Solution

In the next theorem , a stability property of the solution for problem (1) will be substantiated. More precisely, we find an upper bound of $\| \boldsymbol{v(t)} \|_{\boldsymbol{1}} = \| \boldsymbol{v(t)} \|_{\boldsymbol{H^1}}$ that depends on the initial data and the source function.

Theorem 3.1. We assume that $f \in H^1\left([0,T]; L2(\Omega)\right)$, and $v(0) \in H_0^1(\Omega)$ then, $v \in L_\infty\left((0,T); H_0^1(\Omega)\right)$ and

$$\| v(t) \|_1^2 \leq C_1(\| \nabla v(0) \|^2 + \| v(0) \|^2 + \int_0^t \| f'(s) \|^2 \, ds$$

where C1 depends on Ω and $H_0^1(\Omega) = \{v: v , v' \in L_2(\Omega) , \text{u on the boundary equat to zero}\}$

and $L_\infty\left((0,T); H_0^1(\Omega)\right) = \{v: \max_{t\in(0,T)} \| \boldsymbol{v(t)} \|_{\boldsymbol{1}}\}$.

Proof. By taking the inner product of the original problem with *dv/dt*, and use inequality2.7(Green's formula) and integrating over the interval [0, *t*], we obtain

$$\int_0^t \langle {}^C D^{1-\mu} v(s), v'(s)\rangle \, ds + \int_0^t \langle \nabla v(s), \nabla v'(s)\rangle ds = \int_0^t \langle f(s), v'(s)\rangle \, ds \tag{12}$$

By writing $\int_0^t \langle \nabla v(s), \nabla v'(s)\rangle ds$ as

$$\int_0^t \langle \nabla v(s), \nabla v'(s)\rangle ds = \frac{1}{2}\int_0^t \frac{d}{ds} \| \nabla v(s) \|^2 \, ds = \frac{1}{2} \| \nabla v(t) \|^2 - \frac{1}{2} \| \nabla v(0) \|^2$$

We notice that

$$\| \nabla v(t) \|^2 - \| \nabla v(0) \|^2 \leq 2 \left| \int_0^t \langle f(s), v'(s)\rangle \, ds \right|$$

To bound the right hand side of the inequality, let's use integration by parts, then weobtain

$$\int_0^t \langle f(s), v'(s)\rangle \, ds = \langle f\ (t\ ), v(t)\rangle - \langle f\ (0), v(0)\rangle - \int_0^t \langle v(s), f'(s)\rangle \, ds$$

and utilizing Cauchy-Schwarz inequality and the geometric-arithmetic mean inequality ,yields

$$\left| \int_0^t \langle f(s), v'(s)\rangle \, ds \right|$$

$$\leq \| v(t) \| \| f(t) \| + \| v(0) \| \| f(0) \| + \int_0^t \| f'(s)\| \, \| v(s)\| ds \tag{13}$$

$$\leq C \| f(t) \|^2 + \| f(0) \| \| v(0) \| + \frac{1}{4} \| \nabla v(t) \|^2 + C \int_0^t \| f'(s) \|^2 \, ds$$

$$+ \frac{1}{4}\int_0^t \| \nabla v(s) \|^2 \, ds \tag{14}$$

by writing

$$f(t) = f(0) + \int_0^t f'(s) \, ds$$

We obtain

$$\| \nabla v(t) \|^2 \leq \| \nabla v(0) \|^2 + 3 \| f(0) \|^2 + \| v(0) \|^2$$
$$+4 \int_0^t \| f'(s) \|^2 \, ds + \frac{1}{2}\int_0^t \| \nabla v \|^2 \, ds.$$

Hence, using the standard Gronwall's inequality, we obtain

$$\| \nabla v(t) \|^2 \leq \| \nabla v(0) \|^2 + 3 \| f(0) \|^2 + \| v(0) \|^2 + 4 \int_0^t \| f'(s) \|^2 \, ds$$

The proof is completed now.

4- Time-Stepping DPG Method with Finite Element Method

To describe the time-stepping discontinuous Petrov-Galerkin (DPG) method, we introduce nonuniform time partition of the interval [0, T] given by the points: $0 \le t_0 \le t_1 \le ... \le t_N \le T$. We set $= t_n - t_{n-1}$. The maximum step-size is defined as $k = \max_{1\le n\le N} k_n$. The supremum of the function u is defined as

$$\|v(t)\|_{I_n} = sup t \in I_n \ \| v(t) \|.$$

We assume that for a fixed parameter γ there holds

$$c_\gamma k_\gamma \le k_1 \le C_\gamma k_\gamma \tag{15}$$

And

$$c_\gamma k {t_n}^{1-\frac{1}{\gamma}} \le k_n \le C_\gamma k {t_n}^{1-\frac{1}{\gamma}} \tag{16}$$

and $t_n \le C_\gamma t_{n-1}$ for $2 \le n \le N$. For instance, these properties hold if $t_n = (nN)^\gamma T$ for $0 \le n \le N$.

Next, to represent Finite Element Method (FEM) in space for one dimensional model problem , we start by dividing the spatial domain Ω into M_h subintervals of equal length and define the following partition $0 \le x_0 \le x_1 \le \ldots \le x_M$ with $h = x_i - x_{i-1}$, $i = 0,1, \ldots, M_h$. For two dimensional Problem , the domain Ω is divided into smaller triangles to form atriangular mesh. This process is known as triangulation and each triangle is considered as a finite element. Let's denote h_K as the diameter of triangle K with $h = \max_k h_K$. After the that, we define the finite element space S_h as $S_h \subset H^1$ denotes the space of continuous, piecewise linear of degree r($r \ge 1$)with respect to a quasi-uniform partition of Ω into conforming triangular finite elelments, with maximum diameter h. Next, we introduce the trial space

$$W(S_h) = \{v \in C([0, T]; S_h) : v|I_n \in P_1(S_h),\ 1 \le n \le N\}$$

and the test space

$$T(S_h) = \{v \in L_2((0, T); S_h) : v|I_n \in P_0(S_h),\ 1 \le n \le N\}$$

here $P_1(S_h)$ denotes the space of polynomials of degree ≤ 1 in the time variable t, with coefficients in S_h. Now, we are ready to define our numerical scheme. Following [[13], [19]], we define the DPG approximation $V \in W(S_h)$ of the solution v of problem (1) is now defined as follows: Find $V \in W(S_h)$ such that

$$\int_0^T \langle {}^C\mathrm{D}^{1-\mu} V(t), X(t)\rangle + \int_0^T \langle \nabla V(t), \nabla X(t)\rangle\, dt = \int_0^T \langle \mathrm{f(t)}, X(t)\rangle\, dt \tag{17}$$

for all $X(t) \in T(S_h)$ with $V(0) = g(x)$. We notice that the approximate solution $V(x,t)$ is piecewise linear in time with coefficients in S_h.

5- Well-posedness of the fully discrete scheme

In this section, we will show the existence , uniqueness and the stability of the DPG solution. The following theorem proves the existence and uniqueness of the numerical solution.

Theorem 5.1. The numerical solution V of (17) exists and is unique.

Proof. Since the operator $-\Delta$ possesses a complete orthonormal eigensystem $\{\lambda_j, \theta_j\}_{j=1}^{\infty}$ problem (17) can be reduced to a finite linear algebraic equation on each subinterval In. To see this, let P1 be the space of piecewise linear function in time. If we now take $X(t) = \theta_j$ on In and zero elsewhere in 17, then we find that

$$\int_{t_{n-1}}^{t_n} {}^C\mathrm{D}^{1-\mu}\mathrm{V}_j(t) + \lambda_j \mathrm{V}_j(t) dt = \int_{t_{n-1}}^{t_n} f(t) dt \tag{18}$$

for all $j \ge 1$ with $V_j = \langle V, \phi_j\rangle \in P_1$ and $f_j = \langle f, \phi_j\rangle$. Because of the finite dimensionality of system (18) the existence of the scalar function U_j follows from its uniqueness. Since the DPG solution is constructed element by element, it is enough to show the uniqueness on the first time interval $[0,\ t_1]$. That is, it is enough to consider n = 1 in (18) (for n ≥ 2 the proof is completely

analogous). To this end, let V_{j1} and V_{j2} be two DPG solutions on I_1. By linearity, the difference $U_j := \left(V_{j1} - V_{j2}\right)\big|_{I_j}$ then satisfies

$$\int_{t_{n-1}}^{t_n} \left({}^{C}\mathrm{D}^{1-\mu}\mathrm{U}_j + +\lambda_j \mathrm{U}_j\right) dt = 0 \qquad \forall j \geq 1 \tag{19}$$

with with $V^0{}_J = 0$. From 19, we have. From 19, we have

$$\int_0^{t_1} \left({}^{C}\mathrm{D}^{1-\mu}\mathrm{U}_j(t)\right) dt = 0$$

by using lemma 2.4 and $U_j{}^0 = 0$, $U_j{}^1 = 0$, we obtain $U_j = 0$ on $[0, t_1]$.

This completes the proof.

The next theorem shows the stability of the DPG scheme.

Theorem 5.2. Suppose that$g \in \mathrm{L}_2(\Omega)$, and $f \in \mathrm{L}_2(0, \mathrm{T}\,;\ \mathrm{L}2(\Omega))$. Then, for $1 \leq \mathrm{n} \leq \mathrm{N}$, the DPG solution V of (17) satisfies the following property:

$$c_\mu \int_0^T \left\| {}^{C}D^{1-\frac{\mu}{2}} V(t) \right\|^2 \mathrm{dt} + \|\nabla V^n\|^2 \leq \| f(t_n) \|^2 + \| f(0) \| \| g \| + (1 + k_1) \| \nabla g \|^2 + \int_{t_{n-1}}^{t_n} \| f'(t) \|^2 \quad dt$$

Where $c_\mu = \cos(\mu\pi/2)$.

Proof. Taking the inner product of (17)with V ′and zero elsewhere

$$\int_0^{t_n} \langle {}^{C}D^{1-\mu}V(s), V'(s)\rangle\, ds - \int_0^{t_n} \langle \Delta V(s), V'(s)\rangle ds = \int_0^{t_n} \langle f(s), V'(s)\rangle\, ds \tag{20}$$

By using lemma2.3(Green's formula), we obtain

$$\int_0^{t_n} \langle {}^{C}D^{1-\mu}V(s), V'(s)\rangle\, ds + \int_0^{t_n} \langle \nabla V(s), \nabla V'(s)\rangle ds = \int_0^{t_n} \langle f(s), V'(s)\rangle\, ds \tag{21}$$

Following the derivation used to obtain (14), we have

$$\left| \int_0^{t_n} \langle f(s), V'(s)\rangle\, ds \right| \leq \| f\ (t_n)\ \|^{\,2} + \| f(0) \|\ \| V\ (0) \| + \int_0^{t_n} \| f\ '(s) \|^2\ ds + \frac{1}{4} \| \nabla V(t_n) \|^2 + \frac{1}{4} \int_0^{t_n} \| \nabla V(s) \|^2\ ds$$

By using the equality

$$\int_0^{t_n} \langle \nabla V(s), \nabla V'(s)\rangle ds = \frac{1}{2}\ \| \nabla V(t_n) \|^2 - \frac{1}{2}\ \| \nabla V(0) \|^2$$

We obtain

$$2\int_0^{t_n} \langle {}^{C}D^{1-\mu}V(s), V'(s)\rangle ds + \ \| \nabla V(t_n) \|^2$$

$$\leq 2\ \| f\ (t_n)\ \|^{\,2} + 2\ \| f(0) \|\ \| V\ (0) \| + 2\int_0^{t_n} \| f\ '(s) \|^2\ ds + \frac{1}{2} \| \nabla V(t_n) \|^2 + \frac{1}{2} \int_0^{t_n} \| \nabla V(s) \|^2\ ds$$

By using the inequality part(ii) in lemma (2.4) we have,

$$2c_\mu \int_0^T \left\| {}^{C}D^{1-\frac{\mu}{2}} V(t) \right\|^2 \mathrm{dt} + \ \| \nabla V(t_n) \|^2$$

$$\leq 2\ \| f\ (t_n)\ \|^{\,2} + 2\ \| f(0) \|\ \| V(0) \| + 2\int_0^{t_n} \| f\ '(s) \|^2\ ds + \frac{1}{2} \| \nabla V(t_n) \|^2 +$$

$\frac{1}{2}\int_0^{t_n} \| \nabla V(s) \|^2 \, ds$ (22)

However, V_{I_j} is alinear polynomial (in time) , so

$$\| V \|_{I_j} \leq max\left(\left\|V^j\right\|, \left\|V^{j-1}\right\|\right)$$

and hence,

$$\int_0^{t_n} \| \nabla V(t) \|^2 \, dt \leq \sum_{j=1}^{n-1} k_j \left(\left\|\nabla V^j\right\|^2, \left\|\nabla V^{j-1}\right\|^2\right)$$

And hence,

$$\int_0^{t_n} \| \nabla V(t) \|^2 \, dt \leq \sum_{j=1}^{n-1} (k_j + k_{j+1})\left\|\nabla V^j\right\|^2 + k_n\|\nabla V^n\|^2 + k_1\|\nabla V^0\|^2$$

$$\int_0^{t_n} \| \nabla V(t) \|^2 \, dt \leq c_\gamma \sum_{j=1}^{n-1}(k_j)\left\|\nabla V^j\right\|^2 + k_n\|\nabla V^n\|^2 + k_1\|\nabla V^0\|^2 \quad (23)$$

here in the last inequality we shifted the summation indices in the first term and

used the partition assumption $k_{j+1} \leq c_\gamma k_j$, $1 \leq n \leq N$ Insert (23) in (22) and using the assumption k_n is sufficiently small, then for $1 \leq n \leq N$, we obtain

$$2c_\mu \int_0^T \left\|{}^C D^{1-\frac{\mu}{2}} V(t)\right\|^2 \mathrm{dt} + (1 - k_n) \| \nabla V(t_n) \|^2$$

$$\leq 2 \| f(t_n) \|^2 + 2 \| f(0) \| \| V(0) \| + 2\int_0^{t_n} \| f'(s) \|^2 \, ds + 2c_\gamma \sum_{j=1}^{n-1}(k_j)\left\|\nabla V^j\right\|^2$$

(24)

We multiply (24) by $(1 - k_n)^{-1}$, then we get

$$2c_\mu(1 - k_n)^{-1}\int_0^T \left\|{}_t^C D_{0+}^{1-\frac{\mu}{2}} V(t)\right\|^2 \mathrm{dt} + \| \nabla V(t_n) \|^2 \leq 2(1 - k_n)^{-1} \| f(t_n) \|^2 + 2(1 - k_n)^{-1} \| f(0) \| \| V(0) \| + 2(1 - k_n)^{-1}\int_0^{t_n} \| f'(s) \|^2 \, ds + 2c_\gamma(1 - k_n)^{-1}\sum_{j=1}^{n-1}(k_j)\left\|\nabla V^j\right\|^2 \quad (25)$$

From (25), we have

$$\| \nabla V(t_n) \|^2 \leq 2(1 - k_n)^{-1} \| f(t_n) \|^2 + 2(1 - k_n)^{-1}\|f(0)\|\|V(0)\| + 2(1 - k_n)^{-1}\int_0^{t_n} \| f'(s) \|^2 \, ds + 2c_\gamma(1 - k_n)^{-1}\sum_{j=1}^{n-1}(k_j)\left\|\nabla V^j\right\|^2 \quad (26)$$

Thus, by using lemma 2.3, we obtain

$$\| \nabla V(t_n) \|^2 \leq 2(1 - k_n)^{-1} \| f(t_n) \|^2 + 2(1 - k_n)^{-1}\|f(0)\|\|V(0)\| + 2(1 - k_n)^{-1}\int_0^{t_n} \| f'(s) \|^2 \, ds + 2c_\gamma(1 - k_n)^{-1}\exp\left(\sum_{j=1}^{n-1}(k_j)\right)$$

Or

$$\| \nabla V(t_n) \|^2 \leq 2(1 - k_n)^{-1} \| f(t_n) \|^2 + 2(1 - k_n)^{-1}\|f(0)\|\|V(0)\|$$

$$+2(1-k_n)^{-1}\int_0^{t_n} \| f'(s) \|^2\, ds \quad + 2c_\gamma(1-k_n)^{-1}\exp(k(1-k)^{-1}) \qquad (27)$$

Subsituting (27) into (25)

$$4c_\beta(1-k_n)^{-1}\int_0^T \left\| {}^{C}D^{1-\frac{\mu}{2}}V(t)\right\|^2 \mathrm{dt} + \| \nabla V(t_n) \|^2$$
$$\leq 2(1-k_n)^{-1} \| f(t_n) \|^2 + 2(1-k_n)^{-1}\|f(0)\|\|V(0)\| + 2(1-k_n)^{-1}$$
$$\int_0^{t_n} \| f'(s) \|^2\, ds + 2c_\gamma(1-k_n)^{-1}\exp(k(1-k)^{-1}) + 2c_\gamma(1-k_n)^{-1}\exp(k(1-k)^{-1} \qquad (28)$$

by using lemma 2.6, (28) becomes

$$\| V(t) \|^2_{I_n}$$
$$\leq 2(1-k_n)^{-1} \| f(t_n) \|^2 + 2(1-k_n)^{-1}\|f(0)\|\|V(0)\|$$
$$+2(1-k_n)^{-1}\int_0^{t_n} \| f'(s) \|^2\, ds \quad + 2c_\gamma(1-k_n)^{-1}\exp(k(1-k)^{-1}) +$$
$$2c_\gamma(1-k_n)^{-1}\exp(k(1-k)^{-1}\,.$$

This completes the proof.

6- Implementations and numerical results.

This section implements the time-stepping DPG scheme and present asample of numerical results.

6-1 Implementation

In this subsection, we discuss the implementation of the time-stepping DPG scheme. To do this, we define the piecewise linear basis functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ of the finite dimensional trial space $W(S_h)$ as follows: for $j = 1,\ldots, N-1$,

$$\varphi_i(t) = \begin{cases} \frac{t-t_{j-1}}{k_j} & for\quad t \in I_j \\ \frac{t_{j+1}-t}{k_{j+1}} & for\quad t \in I_{j+1} \\ 0 & elsewhere \end{cases}$$

$$\varphi_N(t) = \begin{cases} \frac{t-t_{N-1}}{k_N} & for\quad t \in I_N \\ 0 & elsewhere \end{cases}$$

To go on in our implementation, we reformulate the DPG scheme locally and obtain

$$\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} w_\mu(t-s)\langle V'(s), X(t)\rangle ds\, dt + \int_{t_{n-1}}^{t_n} \langle V(t), X(t)\rangle\, dt$$
$$\int_{t_{n-1}}^{t_n} \langle f(t), X(t)\rangle\, dt - \int_{t_{n-1}}^{t_n}\int_0^{t_{n-1}} w_\mu(t-s)\langle V'(s), X(t)\rangle ds\, dt$$

For all X∈ T and for n=1,2,...N. Using

$V(t) = \sum_{i=0}^{N} a_i\, \emptyset_i(t)$ with a_0=0

And X(t)=1 so, we have to solve

$$\frac{1}{k_n}\langle a_n - a_{n-1}, X(t)\rangle \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} w_\mu(t-s) ds\, dt$$

$$+\frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t}\langle (t_n - t)a_{n-1} + (t-t_{n-1})a_{n-1}, X(t)\rangle\ dt$$

$$= \int_{t_{n-1}}^{t_n} \langle f(t), X(t)\rangle\, dt - \sum_{i=1}^{n-1}\frac{1}{k_i}\langle a_i - a_{i-1}, X(t)\rangle \int_{t_{n-1}}^{t_n}\int_{t_{i-1}}^{t_i} w_\mu(t-s) ds\, dt$$

For n=1,...,N. Integrating

$$\frac{w_{\mu+2}(k_n)}{k_n}\langle a_n - a_{n-1}, X(t)\rangle + \frac{k_n}{2}\langle a_n + a_{n-1}, X(t)\rangle$$

$$= \int_{t_{n-1}}^{t_n}\langle f(t), X(t)\rangle\, dt - \sum_{i=1}^{n-1} w^{n,i}\langle a_i - a_{i-1}, X(t)\rangle \qquad (29)$$

Where

$$w^{n,j} = \frac{1}{k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j} w_\mu\,(t-s)ds\, dt$$

$$= \frac{1}{k_j}\left[w_{\mu+2}(t_{n-1}-t_i) - w_{\mu+2}(t_n - t_i) + w_{\mu+2}(t_n - t_{i-1}) - w_{\mu+2}(t_{n-1}-t_{i-1})\right]$$

Therefore, we arrive the following system

$$2\mathrm{Ba} + \Gamma(\mu + 2)\mathrm{Da} = 2\Gamma(\mu + 2)\mathrm{F} \qquad (30)$$

Where

$$B = \begin{bmatrix} {k_1}^{\mu} & 0 & \cdots & 0 \\ -{k_2}^{\mu} & {k_2}^{\mu} & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & -{k_N}^{\mu} & {k_N}^{\mu} \end{bmatrix},\ D = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ k_2 & k_2 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & k_N & k_N \end{bmatrix},\ a = \begin{bmatrix} \langle a_1, X(t)\rangle \\ \langle a_2, X(t)\rangle \\ \vdots \\ \langle a_N, X(t)\rangle \end{bmatrix},\quad F = \begin{bmatrix} F^1 \\ F^2 \\ \vdots \\ F^N \end{bmatrix}$$

And $F^n = \int_{t_{n-1}}^{t_n}\langle f(t), X(t)\rangle\, dt - \sum_{j=1}^{n-1} w^{n,j}\langle a_i - a_{i-1}, X(t)\rangle$ for $\mathrm{n} = 1,\ldots,\mathrm{N}$.

Next, we let $a_n = \sum_{i=1}^{M_h} a_{n,i}\psi_i$ for $\mathrm{n} = 1,\ldots,\mathrm{N}$. , where ψ_i are basis for S_h for all i = 1, 2, .., M_h and choose

$\mathrm{X} = \psi_j$ for all j = 1, 2, .., n. We subsitute (replace $a_n = \sum_{i=1}^{M_h} a_{n,i}\psi_i$, $\mathrm{X} = \psi_j$) in (1), we get

$$\frac{w_{n+2}(k_n)}{k_n}\langle\sum_{i=1}^{M_h} a_{n,i}\psi_i - \sum_{i=1}^{M_h} a_{n-1,i}\psi_i\ ,\ \psi_j\ \rangle + \frac{k_n}{2}\langle\nabla\sum_{i=1}^{M_h} a_{n,i}\psi_i - \nabla\sum_{i=1}^{M_h} a_{n-1,i}\psi_i\ ,\ \nabla\psi_j\ \rangle$$

$$= \int_{t_{n-1}}^{t_n}\langle f(t), X(t)\rangle\, dt - \sum_{j=1}^{n-1} w^{n,j}\langle\sum_{i=1}^{M_h} a_{n,i}\psi_i - \sum_{i=1}^{M_h} a_{n-1,i}\psi_i\ ,\ \psi_j\ \rangle$$

or

$$\frac{w_{n+2}(k_n)}{k_n}\ A\begin{bmatrix} a_{n,1} - a_{n-1,1} \\ a_{n,2} - a_{n-1,2} \\ \vdots \\ a_{n,M_h} - a_{n-1,M_h} \end{bmatrix} + \frac{k_n}{2}\ E\begin{bmatrix} a_{n,1} + a_{n-1,1} \\ a_{n,2} + a_{n-1,2} \\ \vdots \\ a_{n,M_h} + a_{n-1,M_h} \end{bmatrix}$$

$$= \begin{bmatrix} \int_{t_{n-1}}^{t_n}\langle f(t), \psi_1\rangle\, dt \\ \int_{t_{n-1}}^{t_n}\langle f(t), \psi_2\rangle\, dt \\ \vdots \\ \int_{t_{n-1}}^{t_n}\langle f(t), \psi_{M_h}\rangle\, dt \end{bmatrix} - \sum_{l=1}^{n-1} w^{n,j} A\begin{bmatrix} a_{l,1} - a_{l-1,1} \\ a_{l,2} - a_{l-1,2} \\ \vdots \\ a_{l,M_h} - a_{l-1,M_h} \end{bmatrix}.$$

Where $A = \left[\langle \psi_i , \psi_j \rangle\right]_{M_h \times M_h}$, $E = \left[\langle \nabla\psi_i , \nabla\psi_j \rangle\right]_{M_h \times M_h}$

So, we come to the following system

$$(2A_1 + \Gamma(\mu + 2)E_1)Y = 2\Gamma(\mu + 2)G$$

where

$$A_1 = B \otimes A = \begin{bmatrix} k_1{}^{\mu}A & 0 & \cdots & 0 \\ Ak_2{}^{\mu}A & k_2{}^{\mu}A & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & k_N{}^{\mu}A & k_N{}^{\mu}A \end{bmatrix}$$

$$E_1 = D \otimes E = \begin{bmatrix} k_1E & 0 & \cdots & 0 \\ k_2E & k_2E & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & k_NE & k_NE \end{bmatrix}$$

$$Y = \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{1,M_h} \\ a_{2,1} \\ \vdots \\ a_{2,M_h} \\ \vdots \\ a_{N,1} \\ \vdots \\ a_{N,M_h} \end{bmatrix}, G = \begin{bmatrix} G^1 \\ G^2 \\ \vdots \\ G^N \end{bmatrix}$$

$$G^n = \begin{bmatrix} \int_{t_{n-1}}^{t_n} \langle f(t), \psi_1 \rangle\, dt \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_2 \rangle\, dt \\ \vdots \\ \int_{t_{n-1}}^{t_n} \langle f(t), \psi_{M_h} \rangle\, dt \end{bmatrix} - \sum_{l=1}^{n-1} w^{n,i} A \begin{bmatrix} \langle a_{l,1} - a_{l-1,1}, \psi_1 \rangle \\ \langle a_{l,2} - a_{l-1,2}, \psi_2 \rangle \\ \vdots \\ \langle a_{l,M_h} - a_{l-1,M_h}, \psi_{M_h} \rangle \end{bmatrix}.$$

6-2 Numerical results

In this section, our objective is to validate the accuracy of the error estimates obtained from both the time-stepping DPG scheme and spatial standard finite elements (continuous Galerkin) measured in $L_\infty(0, T; L_2(\Omega))$-norm, for problems of the form (1) when $\Delta u = u_{xx}$ and $\Omega = (0,1)$ and $T = 1$. To evaluate the errors, we introduce the

$$\mathrm{G}^p = \{ t_{j-1} + nk_j/p \ : \ 1 \leq j \leq N,\ 0 \leq n \leq \mathrm{p} \} \qquad (31)$$

(N is the number of time mesh subintervals). Thus, for large values of p, the error measure

$$|||v|||_p = \max_{t \in \mathrm{G}^p} \| v(t) \|$$

approximates the norm $\| v \|_{L\infty}$. To compute the order of convergence with respect to the change in the number of subintervals N , we use the following formula:

$$\frac{\log(\mathrm{error}(\mathrm{N}(\mathrm{i} - 1))/\mathrm{error}(\mathrm{N}(\mathrm{i}))}{\log(\mathrm{N}(\mathrm{i})/\mathrm{N}(\mathrm{i} - 1))}$$

where $N(i) = 2^{i-1}(20)$. We assume that S_h is piece-wise polynomial of degree 1.

Example: We choose the initial datum

$$V(t) = (\pi^2 t^{1-\mu} + \Gamma(2-\mu))cos(\pi x) \quad (32)$$

$$When\ T=1,\ u(t) = t^{1-\mu}\cos(\pi x) \quad (33)$$

The numerical results in Tables 1 ,2 and 3 illustrate the global error bounded by

$$Ck^{\min\{\gamma(1-\mu,2\}}\ \ for\ \gamma \geq 1\ which\ is\ optimal\ \ for\ \gamma \geq 2/(1-\mu)$$

In this phase of the study, we focus on evaluating the performance of the spatial finite elements discretizaton of (order degree 2) of the scheme (17). We use a uniform spatial mesh consists of N subintervals and each is of width h. The time step-size k and the degree of the time-stepping DPG discretization are chosen such that the spatial errors is dominating. Hence, we expect from the numerical results to see convergence of order $O(h^2)$. We illustrated these resultsin tables (1-4).

Table 1. The errors $|||V-v|||_{10}$ for different time mesh gradings (that is, the DPG time stepping solution is piecewise linear) and $\mu = 0.3$. We notice convergence of order $k^{(1-\mu)\gamma}$ $(= k^{0.7\gamma})$ for $1 \leq \gamma \leq (2)/(1-\mu)$ for $v = t^{1-\mu}\cos(\pi x)$.

N	γ	=	1	γ =	2	γ =	3
20	1.7085e-02			3.0361e-03		1.8404e-03	
40	9.3866e-03	8.6402e-01		1.1761e-03	1.3682	4.7905e-04	1.9418
80	5.6166e-03	7.4090e-01		4.5015e-04	1.3855	1.2218e-04	1.9712
160	3.5276e-03	6.7101e-01		1.7161e-04	1.3912	3.0754e-05	1.9902
320	2.2044e-03	6.7834e-01		6.5184e-05	1.3966	7.3289e-06	2.0691

Table 2. The errors $|||V-v|||_{10}$ for for different time partition gradings (that is, the DPG time stepping solution is piecewise linear) and $\mu = 0.5$. We observe the order of convergence is $k^{(1-\mu)\gamma}$ $(= k^{0.5\gamma})$ for $1 \leq \gamma \leq (2)/(1-\mu)$ for $v = t^{1-\mu}\ cos(\pi x)$.

N	γ	=	1	γ =	2.5	γ =	4
20	6.2848e-02			8.7184e-03		2.4507e-03	
40	4.2693e-02	5.5785e-01		3.4681e-03	1.3299	6.3094e-04	1.9576
80	2.8739e-02	5.7099e-01		1.4740e-03	1.2344	1.5997e-04	1.9797
160	1.9271e-02	5.7658e-01		6.2291e-04	1.2427	4.0192e-05	1.9928
320	1.2919e-02	5.7691e-01		2.6249e-04	1.2468	9.7798e-06	2.0390

Table 3. The errors $|||V-v|||_{10}$ for different time mesh gradings (that is, the DPG time stepping solution is piecewise linear) and $\mu = 0.7$. We observe the rate of convergence is $k^{(1-\mu)\gamma}$ $(= k^{0.7\gamma})$ for $1 \leq \gamma \leq (2)/(1-\mu)$ for $u = t^{1-\mu}\ cos(\pi x)$.

N	γ	=	1	γ =	4	γ =	7
10	2.1885e-01			4.7137e-02		2.0476e-02	
20	1.7877e-01	2.9180e-01		1.9397e-02	1.2810	5.2977e-03	1.9505
40	1.4416e-01	3.1047e-01		8.0112e-03	1.2758	1.3284e-03	1.9957
80	1.1576e-01	3.1649e-01		3.3728e-03	1.2481	3.3452e-04	1.9895
160	9.2773e-02	3.1938e-01		1.4427e-03	1.2251	8.4046e-05	1.9928

Table 4. The errors $|||V-v|||_{10}$ for different time mesh gradings with $\mu = 0.3$. We notice the rate of convergenceis h^2

N_x	Error	convergence
4	2.3444e-02	
6	5.8908e-03	1.9927
8	1.4663e-03	2.0063
10	3.3120e-04	2.1464

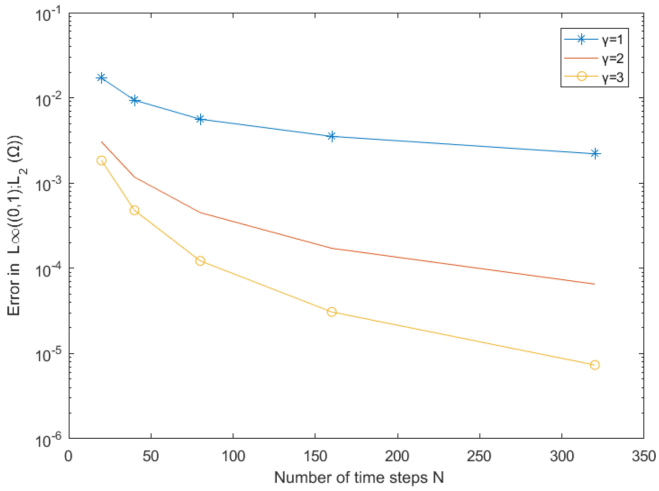

Figure 1. The errors $|||V-v|||_{10}$ plotted against N for different choices of γ, with $\mu = 0.3$ for $v = t^{1-\mu}\cos(\pi x)$.

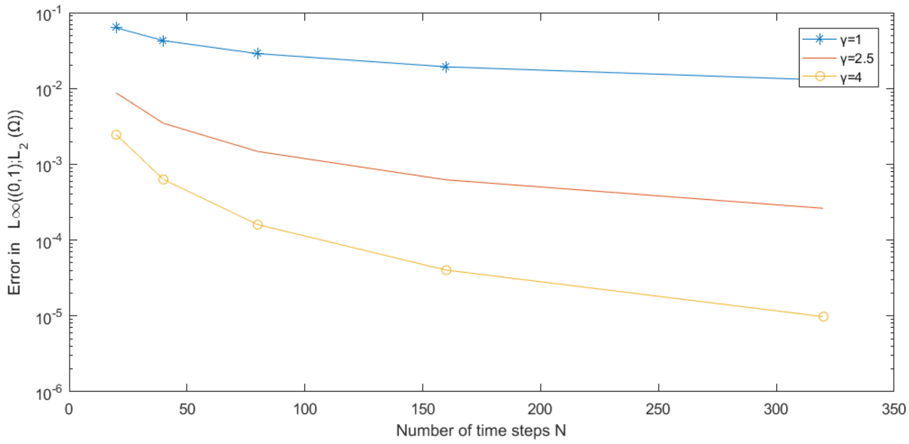

Figure 2. The errors $|||V-v|||_{10}$ for different time mesh gradings plotted against N for different choices of γ, with $\mu = 0.5$ for $v = t^{1-\mu}\cos(\pi x)$.

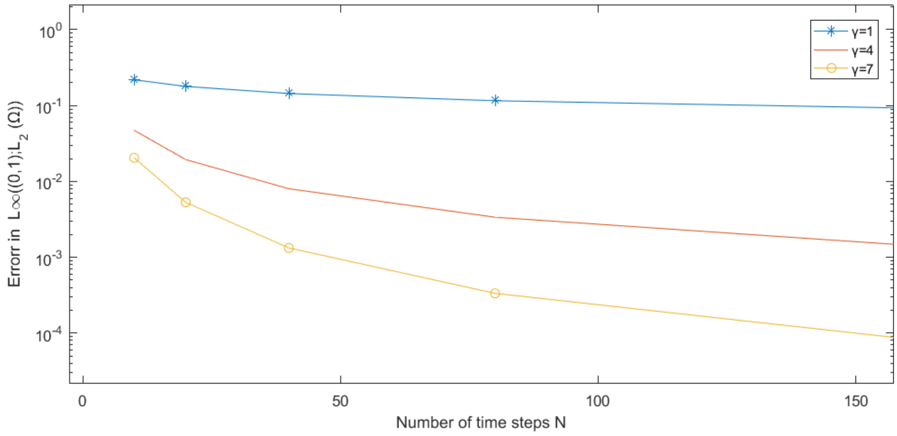

Figure 3. The errors $|||V-v|||_{10}$ plotted against N for different choices of γ, with $\mu = 0.7$ for $v = t^{1-\mu}\cos(\pi x)$.

7- Conclusion

We proved the stability for the exact solution and the approximate solution. For time discretization we used non-uniform mesh and obtained the order of convergence $O(k^{\gamma(1-\mu)})$. For space discretization we used uniform mesh and obtained the order of convergence $O(h^2)$.

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