Arab Journal of Sciences & Research Publishing (AJSRP) • Vol 9, Issue 1 (2023) • P: 108 - 101
https://journals.ajsrp.com/index.php/ajsrp
ISSN: 2518-5780 (Online) • ISSN: 2518-5780 (Print)

Application of Abaoub- Shkheam Transform for Solving Linear Partial Integro – Differential Equations

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Received: 01/12/2022
Revised: 12/12/2022
Accepted: 29/01/2023
Published: 30/03/2023

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Citation: Zali. S. M. (2023). Application of Abaoub- Shkheam Transform for Solving Linear Partial Integro – Differential Equations. Arab Journal of Sciences & Research Publishing, 9 (1),101 – 108. https://doi.org/10.26389/AJSRP.L011222

Abstract: In this paper, we propose a most general form of a linear PIDE with a convolution kernel. We convert the proposed PIDE to an ordinary differential equation (ODE) using a Abaoub- Shkheam - transform (Q). Solving this ODE and applying inverse Abaoub- Shkheam an exact solution of the problem is obtained. It is observed that the Abaoub- Shkheam - transform is a simple and reliable technique for solving such equations. A variety of numerical examples are presented to show the performance and accuracy of the proposed method.

Keywords: Abaoub- Shkheam transform, partial Integro–differential equations, ordinary differential equations.

تطبيق تحويل عبعوب- شخيم لحل المعادلات التفاضلية التكاملية الجزئية الخطية

الدكتورة / سعاد مولود زلي
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المستخلص: نقترح في هذه الورقة الشكل الأكثر عمومية لـ PIDE الخطي مع نواة الالتواء. نقوم بتحويل المعادلات التفاضلية التكاملية الجزئية المقترحة إلى معادلة تفاضلية عادية باستخدام تحويل عبعوب- شخيم ثم حل هذه المعادلة وبتطبيق معكوس تحويل عبعوب شخيم يتم الحصول على حل دقيق للمشكلة. من الملاحظ أن تحويل Q هو تقنية بسيطة وموثوقة لحل مثل هذه المعادلات. يتم تقديم مجموعة متنوعة من الأمثلة العددية لإظهار أداء ودقة الطريقة المقترحة.

الكلمات المفتاحية: تحويل عبعوب – شخيم، المعادلات الجزئية التكاملية، المعادلات التفاضلية العادية.

1- Introduction.

One of the most effective tools for solving problems in physics and engineering is using the transform method to obtain a solution for a given differential equations or integral equation by means of inverse transformation. Among these Transforms are the Fourier [1], Laplace [2], Hankel [3], The importance of an integral transforms is that they provide powerful operational methods for solving initial value problems and initial- boundary value problem for linear differential and integral equations. Recently, in 2013, Khalid S. Aboodh introduces "Aboodh Transform" and applies for solving ordinary differential equations [4-5]. It is also used to solve partial differential equations [6], Heat equations [7].

Real life phenomena are often modelled by ordinary/partial differential equations. Due to the local nature of ordinary differential operator (ODO), the models containing merely ODOs do not help in modelling memory and hereditary properties. One of the best remedies to overcome this drawback is the introduction of integral term in the model. The ordinary/partial differential equation along with the weighted integral of unknown function gives rise to an integro-differential equation (IDE) or a partial integrodifferential equation (PIDE) respectively. Analysis of such equations can be found in [14-22, and application of partial Integro – differential equation play an important role in the various fields of many problems of mathematical fields,

In this article we propose amost general form of a linear PIDE in two independent variables with a convolution kernel.in section 2 we provide some preliminaries regarding QT. section 3 is devoted to the proposed method and section4 provides an number of examples of various types.

2.preliminaries:

2.1-Abaoub- Shkheam Transform "Q – Transform[8]

Definition:Let $f\ (t)$ be a function defined for all $t \geq 0$, the Q-transform of $f\ (t)$ is the function $T\ (u, s)$ defined by

Let $f\ (t)$ be a function defined for all $t \geq 0$, the Q-transform of $f\ (t)$ is the function $T\ (u, s)$ defined by

$$T(u,s) = Q[f] = \int_0^\infty f\ (ut)e^{\frac{-t}{s}}\,dt \ (2.1)$$

$s \in\ (t_1, t_2)$Provided the integral exists for some s, where

The original function $f\ (t)$ in (2.1) is called the inverse transform or inverse of $T(u, s)$, and is denoted by.

$$f(t) = Q^{-1}\{\ T(u,s)\}$$

A list of the Q- transforms for elementaryfunctions ispresented in theTable (1)

Table (1) Q-transformfor someelementaryfunctions

$S.N$	$F\ (t)$	$Q[F\ (t)] = T\ (u,s)$
1.	**1**	s

$S.N$	$F\,(t)$	$Q[F\,(t)] = T\,(u,s)$
2.	t	us^2
3.	t^2	u^2s^3
4.	$t^n \quad s>0, n\in N$	$n!\,u^n s^{n+1}$
5.	e^{at}	$\frac{s}{1-aus}$
6.	$t^r r>-1, s>0$	$\Gamma\,(r+1)u^r s^{r+1}$
7.	$\cos at$	$\frac{s}{1+a^2u^2s^2}$
8.	$\sin at$	$\frac{aus^2}{1+a^2u^2s^2}$
9.	$\cosh at$	$\frac{s}{1-a^2u^2s^2}$
10.	$\sinh at$	$\frac{aus^2}{1-a^2u^2s^2}$

2.2- Convolution transform[11]

Let $f\,(t)$ and $g\,(t)$ having, Abaoub- ShkheamAbaoub- Shkheam s$T_1\,(u,s)$ and $T_2\,(u,s)$, then Abaoub- Shkheam transform of the convolution of f and g is given by

$$[f(t)*g(t)] = uT_1\,(u,s)T_2\,(u,s)$$

Theorem 1:

Abaoub- Shkheam transform of partial derivatives are in the form:

$$1\text{-}Q\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{us}F(x,s,u) - \frac{1}{u}f\,(x,0)$$

$$2\text{ - }Q\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{Qg\,(x,t)}{s} - g(x,0)$$

$$= \frac{1}{us}T(x,s,u) - \frac{1}{u}f(x,0) - s\frac{\partial f}{\partial t}\,(x,0)$$

$$3\text{-}\qquad Q\left[\frac{\partial y}{\partial x}\right] = \frac{d}{dx}T\,(x,s,u)$$

$$4\text{- }Q\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d^2}{dx^2}T\,(x,s,u)$$

proof

To obtain Abaoub- Shkheam transform of partial derivatives, we use integration by parts as follows:

$$1\text{ -}Q\left[\frac{\partial f(x,t)}{\partial t}\right] = \int_0^\infty \frac{\partial}{\partial t} e^{\frac{-t}{s}}\,dt$$

$$\lim_{p\to\infty}\int_0^p \frac{\partial f}{\partial t} e^{\frac{-t}{s}}\,dt = \lim_{p\to\infty} f\,(x,t)e^{\frac{-t}{s}}\,\Big|_0^p + \frac{1}{s}\int_0^p e^{\frac{-t}{s}} f(x,t,u)dt$$

$$= -\frac{1}{\mathrm{u}} v(x,0) + \frac{1}{\mathrm{us}} V \ (2.2)$$

We assume that, f is piecewise continuous and is of exponential order.

Consider,

2- $Q\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]$

$$\text{We have let} \quad \frac{\partial \mathrm{f}}{\partial \mathrm{t}} = g$$

$$Q\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = Q\left[\frac{\partial \mathrm{g(x,t)}}{\partial \mathrm{t}}\right] = Q[g(x,t)] - g\ (x,t)$$

$$\frac{V\ (x,u,s)}{u^2 s^2} - \frac{1}{u^2 \mathrm{s}} v(x,0) - \frac{1}{u}\frac{\partial v}{\partial t}(x,0) \qquad (2.3)$$

Us the Leibnitz' rule (to proof (3) and (4

3- $Q[\frac{\partial f}{\partial t}(x,0)] = \int_0^\infty \mathrm{e}^{\frac{-\mathrm{t}}{\mathrm{s}}} \frac{\partial}{\partial \mathrm{x}} \mathrm{f(x,u,t)dt}$

$$\frac{\partial}{\partial \mathrm{x}} \int_0^\infty \mathrm{e}^{\frac{-\mathrm{t}}{\mathrm{s}}} \mathrm{f(x,u,t)dt} =$$

$$\frac{\mathrm{d}}{\mathrm{dx}} \mathrm{V\ (x,u,s)} =$$

4- $Q\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d^2}{dx^2} V\ (x,u,s)$

We can easily extend this result to the nth partial derivative by using mathematical induction.

3. Solving PIDE using Abaoub- Shkheam transformsmethod:

Now to illustrate the method we consider the general linear partial integral differential equation,

$$\sum_{i=0}^m a_i \frac{\partial^i \mathrm{u}}{\partial \mathrm{t}^i} + \sum_{i=0}^n b_i \frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^r d_i \int_0^t k_i\ (t-s) \frac{\partial^i u(x,s)}{\partial x^i} ds + f(x,t) = 0 \ (3.1)$$

Applying a Abaoub- Shkheam transformation to Equation (3.1), we get

$$\sum_{i=0}^m a_i\, \mathrm{Q}\left[\frac{\partial^i \mathrm{u}}{\partial \mathrm{t}^i}\right] + \sum_{i=0}^n b_i\, Q\left[\frac{\partial^i u}{\partial x^i}\right] + cQ[u] + \sum_{i=0}^r d_i\, Q\left[\int_0^t k_i\ (t-s) \frac{\partial^i u(x,s)}{\partial x^i} ds\right] + Q[f(x,t)] = 0 \ (3.2)$$

Using theorem 1 and theorem 2, we get

$$\sum_{i=0}^m a_i \frac{\mathrm{V\ (x,u,s)}}{u^i s^i} - \frac{1}{u}\sum_{k=0}^{i-1} \frac{v^k \mathrm{(x,0)}}{(us)^{i-k-1}} + \sum_{i=0}^n b_i \frac{d^i \mathrm{V(x,u,s)}}{dx^i} + c\mathrm{V}(x,u,s) + \sum_{i=0}^n d_i\, u\bar{\mathrm{k}}_\iota\mathrm{(u,s)} \frac{d^i \mathrm{V(x,u,s)}}{dx^i} + \bar{\mathrm{f}}(x,0) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)+$$

$$\mathrm{V(x,u,s)} = Q[u(x,t\ )] \quad , \quad \bar{\mathrm{f}}\mathrm{(x,u,s)} = Q[f(x,t)] \text{where}$$

$$\bar{\mathrm{k}}_\iota\mathrm{(u,s)} = Q[\ K_i(u,t)] \qquad \text{and}$$

Equation (3.1) is an ordinary differential equation in $\bar{u}(x$, u, s). Solve this ODE and take inverse Abaoub- Shkheam transform of $\bar{u}(x$, u, s), we get a solution $u\ (x$, u, $t)$ of (3.3)

4-Illustrativeexamples:

In this section we solve first order partial-integro differential Equations and the Second order partial-integro differential equation, wave equation, heat equation, Laplace equation and Telegraphers equation.

In this section we illustrate some examples to explain the presented the method, we chose examples have exact solutions.

Example 4.1:

Consider thewave linear partial integro- differential equation,

$$u_{tt} = u_x + 2\int_0^t (t-s)u\,(x,t,s)\,ds - 2e^x \quad (4.1)$$

with initial condition $u\,(x,0) \;=, u_t\,(x,0) \;= 0$ and boundary condition $u\,(0,t) = cost$

Taking Abaoub- Shkheam transform of Eq. (4.1), we have

$$Q[u_{tt}] = Q[u_x] + 2Q[\int_0^t (t-s)u(x,t,s)\,ds] - 2e^x \mathrm{Q}[1])$$

$$\frac{1}{u^2s^2}V(x,s,u) - \frac{1}{u^2s}v(x,0) - \frac{1}{u}v_t(\mathrm{x},0) = \frac{\mathrm{d}}{\mathrm{dx}}\mathrm{V}(\mathrm{x},\mathrm{s},\mathrm{u}) + 2uQ[x]Q[V(x,s,u) - 2e^x\mathrm{Q}[1]$$

$$\frac{1}{u^2s^2}V(x,s,u) - \frac{1}{u^2s}e^x = V'(x,s,u) + 2\mathrm{u}^2\mathrm{s}^2V(x,s,u) - 2e^x s)$$

$$V'(x,s,u) + \;(2\mathrm{u}^2\mathrm{s}^2 - \frac{1}{u^2s^2})V(x,s,u) = \;(2s - \frac{1}{u^2s})e^x$$

$$I.F = e^{\left(2\mathrm{u}^2\mathrm{s}^2 - \frac{1}{u^2s^2}\right)x}$$

$$V(x,s,u) = \frac{s}{u^2s^2+1}e^x + ce^{\left(\frac{1}{u^2s^2} - 2\mathrm{u}^2\mathrm{s}^2\right)x} \quad (4.2)$$

$$Q[u(0,t) = V(0,t) = Q[\cos t] = \frac{s}{1+u^2s^2} \quad (4.3)$$

Compare (4.3) in (4.2), we get c = 0

$$V(x,s,u) = \frac{s}{u^2s^2+1}e^x$$

Applying inverse Abaoub- Shkheam transform on both sides

$$u(x,t) = Q^{-1}[V(x,s,u) = e^x \cos t$$

Example 4.2: consider theheat partial integro-differential equation

$$u_t - u_{xx} + u + \int_0^t e^{t-s}u(x,s)\,ds = (x^2+1)e^t - 2 \quad (4.4)$$

$$u(x,0) = x^2 \qquad , \quad u_t(x,0) = 1$$

$$u(0,t) = t \qquad , \quad u_x(0,t) = 0$$

Taking Abaoub- Shkheam transform of Eq. (4.4), we have

$$\frac{1}{us}V(x,s,u) - \frac{1}{u}v(x,0) - V''(x,s,u) + V + \frac{us}{1-us}V(x,u,s) = (x^2+1)\frac{s}{1-us} - 2s$$

$$V''(x,s,u)-\left(\frac{1}{us}+\frac{us}{1-us}+1\right)V(x,u,s)=-\frac{1}{u}x^2-(x^2+1)\frac{s}{1-us}+2s$$

$$V''(x,s,u)-\frac{1}{us(1-us)}V(x,u,s)=\frac{-1}{u(1-us)}x^2-\frac{s}{1-us}+2s$$

$$V=C_1e^{\sqrt{\frac{1}{us(1-us)}}x}+C_2e^{-\sqrt{\frac{1}{us(1-us)}}x}+sx^2+us^2$$

Now

$$Q[u\,(0,t)]=V\,(0,t)=us^2$$

$$Q[u_x(0,t)]=V'(0,t\quad)=0$$

Using (17) and (18)in (16) we get,

$C_1+C_2=0$ (4.5)

4.6)($C_1-C_2=0$ and

Solving (4.5) and (4.6) we get, $C_1=\ C_2=0$

$$V=sx^2+us^2$$

Taking inverse Abaoub- Shkheam transform we get,

$$u(x,t)=x^2+t$$

Example 4.3:

Consider the linear partial integro-differential equation

$xu_x\ =\ u_{tt}\ +\ x\sin t\ +\ \int_0^t \sin\,(t\ -\ s)u\,(x,s)ds$ (4.7)

With initial conditions,

$$u\,(x,0)\ =\ 0, u_t\,(x,0)\ =\ x$$

And boundary condition, $u\,(1,t)\ =\ x$

Taking Abaoub- Shkheam transform of Eq. (4.7), we have

$$xV'=\frac{1}{u^2s^2}V(x,s,u)-\frac{1}{u^2s^2}v(x,0)-\frac{1}{u}v_t(\mathrm{x},0)+\frac{us^2x}{1+u^2s^2}+\frac{u^2s^2}{1+u^2s^2}V(x,s,u)$$

$$xV'=\frac{1}{u^2s^2}V(x,s,u)-\frac{1}{u}\mathrm{x}+\frac{us^2x}{1+u^2s^2}+\frac{u^2s^2}{1+u^2s^2}V(x,s,u)$$

$xV'-\left[\frac{1}{u^2s^2}+\frac{u^2s^2}{1+u^2s^2}\right]V(x,s,u)=-\frac{1}{u}\mathrm{x}+\frac{us^2x}{1+u^2s^2}$ (4.8)

Solution of (4.8) is,

$V=us^2x+Cx^{\left[\frac{1}{u^2s^2}+\frac{u^2s^2}{1+u^2s^2}\right]}$ (4.9)

Where C is a constant to be determined, from (4.9), we have,

$$Q[u\,(1,t)]=V\,(1,t)=us^2$$

From (15) C = 0,

then, $V=us^2x$ (4.10)

Taking inverse Q-transform on both the sides of (4.10), we get,

$$u\,(x,t)\;=\;tx$$

5. Conclusions.

In this paper, we have successfully developed the Q- transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and spending a very little time.

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