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Fuzzy Ideals of Pre A^*-Algebras

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Abstract: We've introduced in this paper the notion of fuzzy ideals of Pre A^*–algebras and investigated of some of their properties. We've proved characterization of fuzzy ideals by their level of fuzzy sets. Also, we've proved some characterizations of fuzzy ideals generated by a fuzzy set in terms of their level fuzzy sets too. Furthermore, we've introduced some theorems related to homomorphism of Pre A^*–algebras and images of fuzzy ideals according to them, also some theorems of Cartesian product of fuzzy ideals, and marginal fuzzy set from the Cartesian product of Pre A^*–algebras.

Keywords: Pre A^*-algebra, ideal, principal ideal, homomorphism, fuzzy ideals.

المثاليات الضبابية في جبور Pre A^*

سحر فيصل الحاج ربيع
هيثم عرابي
كلية العلوم || جامعة حلب || سوريا

المستخلص: قدمنا في هذا البحث مفهوم المثاليات الضبابية في جبر Pre A^* ، وتحققنا من بعض خواصها. برهنا ميزات المثاليات الضبابية من خلال مستوى مجموعتها الضبابية. كما أثبتنا بعض صفات المثاليات الضبابية المولَّدة بمجموعة ضبابية من خلال مستوى مجموعتها الضبابية أيضاً.

كذلك قدمنا بعض المبرهنات المتعلقة بهومومورفيزمات جبور Pre A^* وصور المثاليات الضبابية وفقها، وأيضاً بعض المبرهنات المتعلقة بالجداء الديكارتي للمثاليات الضبابية، وهامشية مجموعة ضبابية من جداء ديكارتي لجبور Pre A^*.

الكلمات المفتاحية: جبر Pre A^* ، المثالية، المثالية الرئيسة، الهمومورفيزم، المثاليات الضبابية.

Introduction.

The scientist Birkhoff is considered the first to study the lattice theory in 1989[1]. E.G.Manes introduced the concept of Ada (Algebra of disjoint alternatives) in a paper titled [4]:

"The Equational Theory of Disjoint Alternatives"

The concept was basically based on the extension concept "If- then-else" more than in Boolean algebra.

In 1993, E.G. Manes reintroduced the concept of Ada [5] depending on C-algebra which was defined by F. Guzman and C. Squir in 1990 [2].

In 1994, P. Koteswara Rao introduced the concept of A^* –algebra $(A, \wedge, \vee, *, : , \pi, 0,1,2)$ [3]. He didn't only study the equivalence with Ada, C-algebra, Ada's connection with 3-rings, but also introduced the concept of If- then-else over A^* –algebra and ideal in it.

In 2000, J. Venkateswara Rao introduced the concept of $\mathrm{Pre}\ A^*$ –algebra $(A, \vee, \wedge, :)$ which represents the algebra form of three valued conditional logic [11].

In 2009, Venkateswara Rao et al. generated $\mathrm{Pre}\ A^*$ –algebras from Boolean algebras and defined the congruence relation and Ternary operation on it [7].

In the same year, K. Venkateswara Rao and K. Srinvasa Rao defined a partial ordering on $\mathrm{Pre}\ A^*$ –algebras and studied its properties as a Poset [6].

Boolean algebra represents the algebraic form of two valued conditional logic (true-false), but the concept of Ada ,C , A^*, $\mathrm{Pre}\ A^*$ –algebras are represent algebraic forms of three valued conditional logic (true-false-unknown).

Study problem:

The study problem can be formulated by the following questions:

1- Can the concept of fuzzy groups be used and developed to build a fuzzy ideal in Pre A* - algebra?
2- Is it possible to find characterization of this fuzzy ideal generated by a fuzzy set from the algebraic point of view?
3- What is the relationship between these fuzzy ideals and homomorphisms?
4- How to build a Cartesian product of these fuzzy ideals?

Study supposals:

Our study in this paper will be about the fuzzy ideals in the Pre A*- algebra, introduction of the concept of homomorphisms in the Pre A* - algebra on the fuzzy ideals by finding images of the fuzzy ideals according to these homomorphisms, and introducing the concept of projection and marginality of a fuzzy set from the Cartesian product of the Pre A* - algebra.

Important of the study:

It is a presentation of the definitions of modern partial algebraic structures in the Pre A* - algebra which help in developing this algebra to expand its practical applications, it also expands the horizon and knowledge of those interested in non- classical logic algebras and its applications.

Study methodology.

Collecting information, results, and scientific facts that have been reached in the field of the Pre A* - algebra, analysis of these suppositions and connecting them, deriving new results, and compare it with its similar (if it exists) in other non- classical logic algebras.

1- Preliminaries:

1-1 Definition of Pre A^*-algebra [11]:

Let A a nonempty set supplied with two binary operations "$\vee$" and "$\wedge$", one unary operation "$:$". It is called about the structure $(A,\vee,\wedge,:)$ that it is a Pre A^*–algebra if it satisfies the following statements, for all $x,y,z \in A$:

(Pre A^* 1) $x^{::} = x$

(Pre A^* 2) $x \wedge x = x$

(Pre A^* 3) $x \wedge y = y \wedge x$

(Pre A^* 4) $(x \wedge y)^{:} = x^{:} \vee y^{:}$

(Pre A^* 5) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(Pre A^* 6) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(Pre A^* 7) $x \wedge y = x \wedge (x^{:} \vee y)$

1-2 Example [6]:

Let's take the set A={0,1,2} and supply it with the operations $(\vee,\wedge,:)$ defined in the following tables:

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

x	$x^{:}$
0	1
1	0
2	2

Then the structure $(A,\vee,\wedge,:)$ is a Pre A^*–algebra.

We note that the elements 0,1,2 satisfies the following:

$a)$ $2^{:} = 2$

$b)$ $1 \wedge x = x \qquad \forall x \in A$

$c)$ $0 \vee x = x \qquad \forall x \in A$

$d)$ $2 \wedge x = 2 \vee x = 2 \qquad \forall x \in A$

1-3 Example [6]:

Let's take the set A={0,1} and supply it with the operations $(\vee, \wedge, :)$ defined in the following tables:

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

x	$x^{:}$
0	1
1	0

Then the structure $(A, \vee, \wedge, :)$ is a $\text{Pre } A^*$–algebra, and you can easily satisfy that $(A, \vee, \wedge, :)$ is Boolean algebra.

1-4 Note [8]:

1. The identities $(\text{Pre } A^*\ 1)$ and $(\text{Pre } A^*\ 4)$ imply $\text{Pre } A^*$–algebra with all the dual statements of $(\text{Pre } A^*\ 2)$ to $(\text{Pre } A^*\ 7)$.
2. Every Boolean algebra is a $\text{Pre } A^*$–algebra.
3. A $\text{Pre } A^*$–algebra is a Boolean algebra if it satisfied the two absorption laws.

1-5 Definition of ideal [9]:

Let A a $\text{Pre } A^*$–algebra, and let I a nonempty subset of it.

It is called about I an Ideal if the following conditions were achieved:

$(I_1)\ a, b \in I \ \Rightarrow\ a \vee b \in I$

$(I_2)\ a \in I \ \Rightarrow\ x \wedge a \in I \qquad \forall x \in A$

1-6 Definition of ideal [9]:

Let A a $\text{Pre } A^*$–algebra, and let p an element of it. The set $\{x \wedge p \mid x \in A\}$ is called a principal ideal of A generated by p and is denoted by $< p >$.

1-7 Definition of homomorphism of $\text{Pre } A^*$-algebra [6]:

Let $(A_1, \vee, \wedge, :)$ and $(A_2, \vee, \wedge, :)$ two $\text{Pre } A^*$–algebras. A mapping $f : A_1 \to A_2$ is called a $\text{Pre } A^*$-homomorphism if the following hold, for all $a, b \in A_1$:

1) $f(a \vee b) = f(a) \vee f(b)$

2) $f(a \wedge b) = f(a) \wedge f(b)$

3) $f(a^{:}) = (f(a))^{:}$

1-8 Definition of a partial ordering on $\text{Pre } A^*$-algebra [6]:

Let A a $\text{Pre } A^*$–algebra. The relation "$\le$" defined on A as following: $x \le y \Leftrightarrow x \wedge y = y \wedge x = x$ is a partial ordering on A, for all $x, y \in A$.

1-9 Definition of fuzzy set on a set [10]:

Let X a set. A fuzzy set f of X is a mapping of X into $[0,1]$, that is $f : X \to [0,1]$.

1-10 Definition of contain in fuzzy sets [10]:

Let X a set, and let f, g be two fuzzy sets of X. Then $f \subseteq g$ if $f(x) \le g(x)$, for all $x \in X$.

1-11 Definition of reverse image and direct image to a fuzzy set according to function [10]:

Let X, Y be two sets, and let $f : X \to Y$ be a function.

- For any fuzzy set g in Y, we define $f^{-1}(g)$ as following:

$$\forall x \in X \;;\; (f^{-1}(g))(x) = g(f(x))$$

- For any fuzzy set h in X, we define $f(h)$ as following:

$$\forall y \in Y \;;\; (f(h))(y) = \begin{cases} \max h(x) & ; f(x) = y \\ 0 & ; if\ \ there\ is\ no\ such\ x \end{cases}$$

1-12 Definition of union and intersection of two fuzzy sets [10]:

Let X a set, and let f, g be two fuzzy sets of X. Then for all $x \in X$:

$$(f \cup g)(x) = \max\{f(x), g(x)\}$$
$$(f \cap g)(x) = \min\{f(x), g(x)\}$$

1-13 Definition of Cartesian product of two fuzzy sets [10]:

Let X, Y be two sets, let f be a fuzzy set of X, and g be a fuzzy set of Y. Then for every $(x, y) \in X \times Y$:

$$f \times g : X \times Y \to [0,1]$$
$$(f \times g)(x, y) = \min\{f(x), g(y)\}$$

1-14 Theorem [9]:

Let A a $\mathrm{Pre}\ A^*$–algebra, and let x, p two elements of A. Then: $x \in\ <p> \Leftrightarrow x = x \wedge p$

1-15 Theorem [9]:

Let A a $\mathrm{Pre}\ A^*$–algebra, and let X be a nonempty set on it. Then the set $\{\vee_{i=1}^{n}(a_i \wedge x_i) \mid a_i \in A, x_i \in X\}$ is the smallest ideal containing X, and is denoted by $<X>$.

1-16 Theorem [6]:

Let A a $\text{Pre } A^*$–algebra. Then a Poset $(A,\leq)$ with 1, for any $x,y\in A$,
$\inf\{x,y\}=x\wedge y$.

2- Research Results.

2-1 Definition:

Let A a $\text{Pre } A^*$–algebra, and let f be a fuzzy set of A. For any $t\in[0,1]$, the set $f_t=\{x\in A \mid f(x)\geq t\}$ is called a level fuzzy set f at t.

- Since $f(x)\in[0,1]$, it is clear from the previous definition that
  $f(x)=\max\{t\in[0,1] \mid x\in f_t\}$.

2-2 Definition:

Let A a $\text{Pre } A^*$–algebra, and let f be a fuzzy set of A. We called f a fuzzy ideal of A, if the following hold, for any $x,y\in A$ 1) $f(x\vee y)\geq f(x)\wedge f(y)$

2) $f(x\wedge y)\geq f(y)$ *or* $f(x\wedge y)\geq f(x)$

- It is clear that $f(x\vee y)=f(y\vee x)$, because A is a $\text{Pre } A^*$–algebra.

2-3 Lemma:

Let A a $\text{Pre } A^*$–algebra, and let f be a fuzzy ideal of A. For any $a,b\in A$, the following hold:

1) $if\ x\in\langle a\rangle\Rightarrow f(x)\geq f(a)$

2) $if\ x\in\langle S\rangle\Rightarrow f(x)\geq \overset{n}{\underset{i=1}{\wedge}} f(a_i)\ \ ;a_1,a_2,\ldots,a_n\in S$

Proof:

1- Since $x\in\langle a\rangle$ then it is according to (1-14) is $x=x\wedge a$, then $f(x)=f(x\wedge a)\geq f(a)$.

2- Since $x\in\langle S\rangle$ then $x=\overset{n}{\underset{i=1}{\vee}}(a_i\wedge s_i)\ ;a_i\in A, s_i\in S$, and since f is a fuzzy ideal of A, then:

$$f(x)=f(\overset{n}{\underset{i=1}{\vee}}(a_i\wedge s_i))\geq \overset{n}{\underset{i=1}{\wedge}} f(a_i\wedge s_i)\geq \overset{n}{\underset{i=1}{\wedge}} f(a_i)$$

2-4 Corollary:

Let A a $\text{Pre } A^*$–algebra. Then every fuzzy ideal f of A is an inversor order mapping.

Proof:

Let $x,y\in A$, where $x\leq y$, then $x\wedge y=x$, consequently $f(x\wedge y)=f(x)$.

Since f is a fuzzy ideal of A, according to the second condition, we find $f(x \wedge y) \geq f(y) \Rightarrow f(x) \geq f(y)$.

2-5 Definition:

Let A a $\text{Pre } A^*$–algebra, and let f be a fuzzy set of A. The smallest fuzzy ideal of A containing f will be called a fuzzy ideal Of A generated by f, and is denoted by i_f.

2-6 Theorem:

Let A a $\text{Pre } A^*$–algebra, and let f be a fuzzy set of A. Then f is a fuzzy ideal of A, if and only if each level of a fuzzy set f is an ideal of A.

Proof:

$(\Leftarrow)$ Let's suppose f is a fuzzy ideal of A, and let $x, y \in f_t$.

- Since $x, y \in f_t$, then $f(x) \geq t$ & $f(y) \geq t$, then

$$f(x \vee y) \geq f(x) \wedge f(y) \geq t \wedge t = t \Rightarrow x \vee y \in f_t$$

- Since $x \in f_t$, $a \in A$, then $f(x) \geq t$, then

$$f(a \wedge x) \geq f(x) \geq t \Rightarrow a \wedge x \in f_t$$

So f_t is an ideal of A, for all $t \in [0,1]$.

$(\Rightarrow)$ Let's suppose each level of a fuzzy set f is an ideal of A, then:

- For any $x, y \in A$, and according to definition of the level fuzzy set (1-2), we find:

$$f(x) \wedge f(y) = \max\{t \in [0,1] \mid x \in f_t\} \wedge \max\{u \in [0,1] \mid y \in f_u\}$$
$$= \max\{\min\{t, u\} \in [0,1] \mid x \in f_t, y \in f_u\}$$

Suppose $\min\{t, u\} = t$. Since $y \in f_u$, then $y \in f_t$, because

$$y \in f_u \Rightarrow f(y) \geq u \;\&\; u \geq t \Rightarrow f(y) \geq t \Rightarrow y \in f_t$$

Then $x \vee y \in f_t$ (because f_t is an ideal of A), so

$$f(x) \wedge f(y) \leq \max\{t \in [0,1] \mid x \vee y \in f_t\} = f(x \vee y)$$

The first condition of the fuzzy ideal definition is hold.

- Let $x, y \in A$, and suppose $t \in [0,1]$, where $y \in f_t$, since f_t is an ideal of A, then $x \wedge y \in f_t$, so

$$f(y) = \max\{t \in [0,1] \mid y \in f_t\} \leq \max\{t \in [0,1] \mid x \wedge y \in f_t\} = f(x \wedge y)$$
$$= f(x \wedge y)$$

The second condition of the fuzzy ideal definition is hold.

2-7 Property:

Let A a $\text{Pre } A^*$–algebra, and let $a, b, c, d \in A$, then:

$$a \le b \ \& c \le d \Rightarrow a \vee c \le b \vee d$$

Proof:

$$a \le b \ \& c \le d \Rightarrow a \wedge b = a \ \& c \wedge d = c$$

We have:

$$\begin{aligned} a \vee c &= (a \wedge b) \vee (c \wedge d) = (c \wedge d) \vee (a \wedge b) \\ &= ((c \wedge d) \vee a) \wedge ((c \wedge d) \vee b) = (a \vee (c \wedge d)) \wedge (b \vee (c \wedge d)) \\ &= (a \vee (c \wedge d)) \wedge (b \vee c) \wedge (b \vee d) = (a \vee c) \wedge (b \vee c) \wedge (b \vee d) \\ &= ((a \wedge b) \vee c) \wedge (b \vee d) = (a \vee c) \wedge (b \vee d) \Rightarrow a \vee c \le b \vee d \end{aligned}$$

2-8 Definition:

Let A a $\mathrm{Pre}\ A^*$ –algebra, and let f be a fuzzy set of A. For any $x \in A$:

$$\hat{f}(x) = \max\left\{ t \in [0,1] \middle| \ x \le a \wedge b; a \in A, b \in f_t \right\}$$

2-9 Theorem:

Let A a $\mathrm{Pre}\ A^*$ –algebra, and let f be a fuzzy ideal of A. Then $\hat{f} = i_f$:

Proof:

Let $x, y \in A$, we have:

- $$\hat{f}(x) \wedge \hat{f}(y) = \max\left\{ t \in [0,1] \middle| \ x \le a \wedge b; a \in A, b \in f_t \right\} \wedge \max\left\{ u \in [0,1] \middle| \ y \le a' \wedge b'; a' \in A, b' \in f_u \right\}$$

If we suppose $\min\{t, u\} = t$, then we have:

$$b' \in f_u \Rightarrow f(b') \ge u \ge t \Rightarrow b' \in f_t$$

There for:

$$\begin{aligned} \hat{f}(x) \wedge \hat{f}(y) &= \max\{ t \in [0,1] | \ x \vee y \le (a \wedge b) \vee (a' \wedge b') \\ &\qquad ; a, a' \in A, b, b' \in f_t \} \\ &\le \max\{ t \in [0,1] | \ x \vee y \le (a \vee a') \wedge (a \vee b') \wedge \\ &\qquad (b \vee a') \wedge (b \vee b'); a \vee a' \in A, b \vee b' \in f_t \} \\ &= \max\{ t \in [0,1] | \ x \vee y \le (a \vee a') \wedge (b \vee b') \\ &\qquad ; a \vee a' \in A, b \vee b' \in f_t \} \\ &= \hat{f}(x \vee y) \end{aligned}$$

- $\hat{f}(y) = \max\left\{ t \in [0,1] \middle| \; y \le a \wedge b; a \in A, b \in f_t \right\}$

$$= \max\left\{ t \in [0,1] \middle| \; x \wedge y \le y \le a \wedge b; a \in A, b \in f_t \right\}$$

$$= \hat{f}(x \wedge y)$$

Hence $\hat{f}$ is a fuzzy ideal of A. Let prove now that $f \subseteq \hat{f}$:

$$\forall x \in f_t \; ; f(x) = \max\left\{ t \in [0,1] \middle| \; x \in f_t \right\}$$

$$= \max\left\{ t \in [0,1] \middle| \; x \le x \wedge x ; x \in A, x \in f_t \right\}$$

$$= \hat{f}(x)$$

So $f(x) \le \hat{f}(x)$.

Let h be a fuzzy ideal of A, where $f \subseteq h$, then $f_t \subseteq h_t$, because:

$$\forall x \in f_t \Rightarrow f(x) \ge t \;\; \& \; h(x) \ge f(x)$$

$$\Rightarrow h(x) \ge t \Rightarrow x \in h_t$$

Since h_t is an ideal of A, for all $t \in [0,1]$ (according to theorem 2-6) and containing f_t, we have:

- $\hat{f}(x) = \max\left\{ t \in [0,1] \middle| \; x \le a \wedge b; a \in A, b \in f_t \right\}$

$$= \max\left\{ t \in [0,1] \middle| \; x \le a \wedge b; a \in A, b \in f_t \subseteq h_t \right\}$$

$$\le \max\left\{ t \in [0,1] \middle| \; x \le a \wedge b; a \wedge b \in h_t \right\}$$

$$= \max\left\{ t \in [0,1] \middle| x \in h_t \right\}$$

$$= h(x)$$

Hence $\hat{f} \subseteq h$. So $\hat{f}$ is a smallest fuzzy ideal of A containing f.

2-10 Theorem:

Let A, B be two $\operatorname{Pre} A^*$–algebras, let $K : A \to B$ be $\operatorname{Pre} A^*$ epimorphism, and let f be a fuzzy ideal of A and g be a fuzzy ideal of B. Then:

1) $K(f)$ be a fuzzy ideal of B.

2) $K^{-1}(g)$ be a fuzzy ideal of A.

Proof:

1- Let $b_1, b_2 \in B$, and suppose that $a_i \in A$, where $K(a_i) = b_1 ; i \in I$ and $a_j \in A$, where $K(a_j) = b_2 ; j \in J$. Then, according to the definition of (11-1), we find:

- $(K(f))(b_1) \wedge (K(f))(b_2) = \max\{f(a_i) | a_i \in A; K(a_i) = b_1; i \in I\} \wedge$
$\max\{f(a_j) | a_j \in A; K(a_j) = b_2; j \in J\}$

$= \max\{f(a_i) \wedge f(a_j) | a_i, a_j \in A; K(a_i) = b_1, K(a_j) = b_2; i \in I, j \in J\}$

$\leq \max\{f(a_i \vee a_j) | a_i \vee a_j \in A; K(a_i \vee a_j) = K(a_i) \vee K(a_j) = b_1 \vee b_2; i \in I, j \in J\}$

$= (K(f))(b_1 \vee b_2)$

- $(K(f))(b_2) = \max\{f(a_j) | a_j \in A; K(a_j) = b_2; j \in J\}$

$\leq \max\{f(a_i \wedge a_j) | a_i, a_j \in A; K(a_i) = b_1, K(a_j) = b_2; i \in I, j \in J\}$

$= \max\{f(a_i \wedge a_j) | a_i \wedge a_j \in A; K(a_i \wedge a_j) = K(a_i) \wedge K(a_j) = b_1 \wedge b_2; i \in I, j \in J\}$

$= (K(f))(b_1 \wedge b_2)$

Thus $K(f)$ be a fuzzy ideal of B .

2) Let $a_1, a_2 \in A$, for any $b_1, b_2 \in B$, we find:

- $(K^{-1}(g))(a_1) \wedge (K^{-1}(g))(a_2) = g(K(a_1)) \wedge g(K(a_2))$
$\leq g(K(a_1) \vee K(a_2))$
$= g(K(a_1 \vee a_2))$
$= (K^{-1}(g))(a_1 \vee a_2)$

- $(K^{-1}(g))(a_2) = g(K(a_2)) \leq g(K(a_1) \wedge K(a_2))$
$= g(K(a_1 \wedge a_2)) = (K^{-1}(g))(a_1 \wedge a_2)$

Thus $K^{-1}(g)$ be a fuzzy ideal of A .

2-11 Note:

Let A, B be two sets, let $K : A \to B$ be a mapping. For any two fuzzy sets f, g of A and two fuzzy sets h, i of B . Then:

1) $f \subseteq g \Rightarrow K(f) \subseteq K(g)$

2) $h \subseteq i \Rightarrow K^{-1}(h) \subseteq K^{-1}(i)$

Proof:

1) $\forall b \in B$; $(K(f))(b) = \max\{f(a_i) | K(a_i) = b, \forall a_i \in A, i \in I\}$

Since $f \subseteq g$, so $f(a_i) \leq g(a_i)$, then:

$(K(f))(b) \leq \max\{g(a_i) | K(a_i) = b, \forall a_i \in A, i \in I\} = (K(g))(b)$

But if $K(a_i) \neq b$, then $(K(f))(b) = 0$, and in this case it will be also $(K(g))(b) = 0$. Thus $K(f) \subseteq K(g)$

2) $\forall a \in A;\ (K^{-1}(h))(a) = h(K(a))$

Since $h \subseteq i$, then $h(b) \leq i(b)\ ; b \in B$, so:

$(K^{-1}(h))(a) \leq i(K(a)) = (K^{-1}(i))(a) \Rightarrow K^{-1}(h) \subseteq K^{-1}(i)$

2-12 Theorem:

Let A, B be two $\mathrm{Pre}\ A^*$-algebras, let $K: A \to B$ be $\mathrm{Pre}\ A^*$ epimorphism, and let f, g be two fuzzy ideals of A and h, i be two fuzzy ideals of B. Then:

1) $K(f \cup g) = K(f) \vee K(g) \qquad ; f \subseteq g$

2) $K(f \cap g) = K(f) \wedge K(g)$

3) $K^{-1}(h \cup i) = K^{-1}(h) \vee K^{-1}(i) \qquad ; f \subseteq g$

4) $K^{-1}(h \cap i) = K^{-1}(h) \wedge K^{-1}(i)$

Proof:

- $K(f \cup g)$ is a fuzzy ideal of B according to the first item from the previous theorem.
- For any $b \in B$, then:
- $(K(f))(b) = \max\{f(a_i) | a_i \in A; K(a_i) = b; i \in I\}$

$$\leq \max\{(f \cup g)(a_i) | a_i \in A; K(a_i) = b; i \in I\}$$
$$= (K(f \cup g))(b)$$
$$\Rightarrow K(f) \subseteq K(f \cup g)$$

Similarly, we find $K(g) \subseteq K(f \cup g)$

Hence $K(f \cup g)$ is an upper bound of $K(f), K(g)$ in $[0,1]$.

- Let j be another fuzzy ideal of B, where $K(f) \subseteq j\ \&\ K(g) \subseteq j$.

$\Rightarrow K^{-1}(K(f)) \subseteq K^{-1}(j)\ \&\ K^{-1}(K(g)) \subseteq K^{-1}(j)$

$\Rightarrow f \subseteq K^{-1}(j)\ \&\ g \subseteq K^{-1}(j) \Rightarrow f \cup g \subseteq K^{-1}(j)$

$\Rightarrow K(f \cup g) \subseteq K(K^{-1}(j)) \Rightarrow K(f \cup g) \subseteq j$

Hence $K(f \cup g)$ is the smallest fuzzy ideal of B containing $K(f)$ and $K(g)$.

The others can be proved using similar arguments.

2-13 Theorem:

Let A, B be two $\mathrm{Pre}\ A^*$-algebras, and let f, g be two fuzzy ideals of A and B respectively. Then $f \times g$ is a fuzzy ideal of $A \times B$.

Proof:

Let $(a_1,b_1),(a_2,b_2)\in A\times B$. We have:

- $(f\times g)((a_1,b_1)\vee(a_2,b_2))=(f\times g)(a_1\vee a_2,b_1\vee b_2)$

$$
\begin{aligned}
&=\min\left\{f(a_1\vee a_2),g(b_1\vee b_2)\right\}\\
&\geq\min\left\{\min\{f(a_1),f(a_2)\},\min\{g(b_1),g(b_2)\}\right\}\\
&=\min\left\{\min\{f(a_1),g(b_1)\},\min\{f(a_2),g(b_2)\}\right\}\\
&=\min\left\{(f\times g)(a_1,b_1),(f\times g)(a_2,b_2)\right\}\\
&=(f\times g)(a_1,b_1)\wedge(f\times g)(a_2,b_2)
\end{aligned}
$$

- $(f\times g)((a_1,b_1)\wedge(a_2,b_2))=(f\times g)(a_1\wedge a_2,b_1\wedge b_2)$

$$
\begin{aligned}
&=\min\left\{f(a_1\wedge a_2),g(b_1\wedge b_2)\right\}\\
&\geq\min\left\{f(a_2),g(b_2)\right\}\\
&=(f\times g)(a_2,b_2)
\end{aligned}
$$

2-14 Definition:

Let A,B be two $\mathrm{Pr}e\ A^*$–algebras, and let f be a fuzzy set of $A\times B$. A fuzzy set $pr_A(f):A\to[0,1]$, which is defined as:

$$\forall a\in A\ ;\ pr_A(f)(a)=\max\{f(a,b)\,|\,b\in B\}$$

is called the projection of f on A . Similarly we can define the projection of f on B as follow:

$$\forall b\in B\ ;\ pr_B(f)(b)=\max\{f(a,b)\,|\,a\in A\}$$

2-15 Theorem:

Let A,B be two $\mathrm{Pr}e\ A^*$–algebras, and let f be a fuzzy ideal of $A\times B$. Then $pr_A(f)$ ($pr_B(f)$) be a fuzzy ideal of A (B).

Proof:

Let $a_1,a_2\in A$. We have:

- $pr_A(f)(a_1) \wedge pr_A(f)(a_2) = \max\{f(a_1,b_i) \mid b_i \in B; i \in I\} \wedge$
$$\max\{f(a_2,b_j) \mid b_j \in B; j \in J\}$$
$$= \max\{f(a_1,b_i) \wedge f(a_2,b_j) \mid b_i, b_j \in B; i \in I, j \in J\}$$
$$\leq \max\{f((a_1,b_i) \vee (a_2,b_j)) \mid b_i, b_j \in B; i \in I, j \in J\}$$
$$\leq \max\{f(a_1 \vee a_2, b_i \vee b_j) \mid b_i \vee b_j \in B; i \in I, j \in J\}$$
$$= pr_A(f)(a_1 \vee a_2)$$
- $pr_A(f)(a_2) = \max\{f(a_2,b_j) \mid b_j \in B; j \in J\}$
$$\leq \max\{f((a_1,b_i) \wedge (a_2,b_j)) \mid a_1 \in A; b_i, b_j \in B; i \in I, j \in J\}$$
$$= \max\{f(a_1 \wedge a_2, b_i \wedge b_j) \mid a_1 \in A; b_i, b_j \in B; i \in I, j \in J\}$$
$$\leq \max\{f(a_1 \wedge a_2, b_i \wedge b_j) \mid a_1 \in A; b_i \wedge b_j \in B; i \in I, j \in J\}$$
$$= pr_A(f)(a_1 \wedge a_2)$$

Then $pr_A(f)$ be a fuzzy ideal of A .

Similarly, we can prove that $pr_B(f)$ be a fuzzy ideal of B .

2-16 Definition:

Let A, B be two $\mathrm{Pre}\ A^*$ –algebras, and let f be a fuzzy set of $A \times B$, and suppose that $a \in A$ and $b \in B$. We can define the fuzzy set $f_A^{(b)} : A \to [0,1]$ as follow:

$\forall x \in A$; $f_A^{(b)}(x) = f(x,b)$ and call it the marginal fuzzy set of f with respect to b . And can define the fuzzy set $f_B^{(a)} : B \to [0,1]$ as follow: $\forall y \in B$; $f_B^{(a)}(y) = f(a,y)$ and call it the marginal fuzzy set of f with respect to a .

2-17 Theorem:

Let A, B be two $\mathrm{Pre}\ A^*$ –algebras, and let f be a fuzzy ideal of $A \times B$, and suppose that $a \in A$ and $b \in B$. Then $f_A^{(b)}$ ($f_B^{(a)}$) be a fuzzy ideal of A (B).

Proof:

Let $x_1, x_2 \in A_1$. We have:

- $f_A^{(b)}(x_1) \wedge f_A^{(b)}(x_2) = f(x_1,b) \wedge f(x_2,b)$
$$\leq f((x_1,b) \vee (x_2,b))$$
$$= f(x_1 \vee x_2, b \vee b)$$
$$= f(x_1 \vee x_2, b) = f_A^{(b)}(x_1 \vee x_2)$$

- $f_A^{(b)}(x_2) = f(x_2, b) \leq f((x_1, b) \wedge (x_2, b))$
  $= f(x_1 \wedge x_2, b \wedge b) = f(x_1 \wedge x_2, b)$
  $= f_A^{(b)}(x_1 \wedge x_2)$

So $f_A^{(b)}$ be a fuzzy ideal of A. Similarly we can prove that $f_B^{(a)}$ be a fuzzy ideal of B.

Recommendations.

This paper lays the basis of further studies on the fuzzy ideal theories in the Pre A* - algebras like: Prime fuzzy ideals, relationship between fuzzy ideals and prime fuzzy ideals, fuzzy congruence relation, create a lattice of fuzzy ideals, and other fuzzy structures in the Pre A* - algebra.

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